Physics 407-07
Assignment 1

1. Given a function $f(p)$ and a set of coordinates $x^i(p)$ show that the two functions

$$f(p) \tag{1}$$
$$\sum_i \frac{\partial f(p(\mathbf{x}))}{\partial x^i}|_{p_0}(x^i(p) - x^i(p_0)) \tag{2}$$

have the same cotangent vector at the point p_0. (p(x) is the point p in the space corresponding to the coordinates $\mathbf{x}$. Those partial derivatives are evaluated at the point p_0. Since $\frac{\partial f(p(\mathbf{x}))}{\partial x^i}|_{p_0}$ are constants, by the definition of the sum of cotangent vectors, this means that

$$df_A = \sum_i \frac{\partial f(p(\mathbf{x}))}{\partial x^i}|_{p_0} dx^i_A \tag{3}$$

****

The two have the same cotangent vector is for any nice curve $\gamma(\lambda)$ the derivatives along the curve are the same.

We can write

$$f(\gamma(\lambda)) = f(P(x^i(\gamma(\lambda))) \tag{4}$$

and

$$\frac{d}{d\lambda} f(\gamma(\lambda)) = \sum_i \partial_i f(P(x^i(\gamma(\lambda))) \frac{dx^i(\gamma(\lambda))}{d\lambda} \tag{5}$$

while

$$\frac{d}{d\lambda}(\sum_i \partial_i (f(P(x^i))(x^i(\gamma(\lambda) - x^i(p_0)) = \sum_i \partial_i (f(P(x^i)) \frac{dx^i(\gamma(\lambda))}{d\lambda} \tag{6}$$

which is the same. Thus for all nice curves the two functions have the same derivative at p_0 along the curve. Thus

$$df_A = \sum_i \partial_i f(P(x^i)) dx^i_A \tag{7}$$

---

2. Show that if x^i and $\tilde{x}^i$ are two different coordinates , and $\gamma(\lambda)$ and $\gamma'(\lambda)$ are two different curves through the point p_0 with the point p_0 corresponding to the same value, $\lambda = 0$ in both cases, that the two curves defined by

$$\Gamma(\lambda) = p(x^i(\gamma(\lambda)) + x^i(\gamma'(\lambda)) - x^i(p_0)) \tag{8}$$
$$\tilde{\Gamma}(\lambda) = p(\tilde{x}^i(\gamma(\lambda)) + \tilde{x}^i(\gamma'(\lambda)) - \tilde{x}^i(p_0)) \tag{9}$$

have the same tangent vector at the point p_0

** Consider any function f(p). Then

$$\frac{d}{d\lambda} f(P(x^i(\gamma(\lambda)) + x^i(\gamma'(\lambda) - x^i(p_0))) = \sum_i \partial_i(f(P(x^i)))|_{x^i=x^i(p_0)} \frac{d}{d\lambda}\left(x^i(\gamma(\lambda)) + x^i(\gamma'(\lambda)) - x^i(p_0)\right) \tag{10}$$

$$= \sum_i \partial_i(f(P(x^i)))|_{x^i=x^i(p_0)} \left(\frac{dx^i(\gamma(\lambda))}{d\lambda} + \frac{dx^i(\gamma(\lambda))}{d\lambda}\right) \tag{11}$$

$$= \frac{df(x^i(\gamma(\lambda))}{d\lambda} + \frac{df(P(x^i(\gamma'(\lambda))))}{d\lambda} = \frac{df(\gamma(\lambda))}{d\lambda} + \frac{df(\gamma'(\lambda))}{d\lambda} \tag{12}$$

Ie, no matter what the coordinate system, the derivative of any function along that sum curve is the sum of the derivatives along the two curves. This is independent of which coordinate system we choose, depending only $f(P(x^i))$ being a differentiable function of the coordinates.

———————————————————————————

This shows that while the definition of the addition of two tangent vectors is defined via coordinates, the sum tangent vector thus defined does not depend on which coordinates we use, although the curves Γ and $\tilde{\Gamma}$ are in general different.

As an example, consider the two curves in two dimensions with coordinates x,y and r, θ

$$\gamma: \tag{13}$$
$$x = \lambda \tag{14}$$
$$y = 1 \tag{15}$$
$$\gamma': \tag{16}$$
$$y = 1 + 2\lambda \tag{17}$$
$$x = 0 \tag{18}$$

Now write those same two curves in terms of the coordinates r, θ where

$$x = r\cos(theta) y = r\sin(theta) \tag{19}$$

Show that the sum curve $\Gamma(\lambda)$ defined in the two coordinate systems differ, but that at the point $\lambda = 0$ their tangent vectors do not.

** ******************************

The sum curve in x y coordinates is

$$x = \lambda \tag{20}$$
$$y = 1 + 2\lambda \tag{21}$$

In r θ coordinates, we have γ:

$$r = \sqrt{\lambda^2 + 1} \tag{22}$$
$$\theta = atan(\frac{1}{\lambda}) \tag{23}$$

while γ':

$$r = |(1 + 2\lambda)| \tag{24}$$
$$\theta = sign(1 + 2\lambda)\pi/2 \tag{25}$$

The sum of these two curves is thus

$$r = \sqrt{\lambda^2 + 1} + |(1 + 2\lambda)|1 \tag{26}$$
$$\theta = \theta = atan(\frac{1}{\lambda}) + sign(1 + 2\lambda)\pi/2 - \pi/2 \tag{27}$$

Writing these in x, y coordinates, we have

$$x = (\sqrt{\lambda^2 + 1} + |(1 + 2\lambda)|1)(\frac{\lambda}{\sqrt{\lambda^2 + 1}}) \tag{28}$$
$$y = (\sqrt{\lambda^2 + 1} + |(1 + 2\lambda)|\frac{1}{\sqrt{\lambda^2 + 1}}) \tag{29}$$

This is certainly not the same as the first sum curve. The common point is x=0, y=2.

———————————————————————

3. Assume that ${H^A}_B$, ${L_A}^B{}_C$ and M_{AB} are tensors, and f, g are functions. Which of the following are tensors and why?

i)${Q_A}^B = {H^B}_A$

****

This is a tensor. The function Q of a tangent and cotangent vector is a linear function of those vectors. It is just that the order of the function arguments of Q is different from that of H.

———————————————————————

ii)$R = {H^A}_A$

****

This is a tensor. It is a function of no vector arguments. The operation of contraction is defined for tensors and is a valid tensor operation.

———————————————————————-

iii)$T^D_{ABC} = {H^D}_A M_{BC}$

****

The function on the left is a function of the same arguments as on the right (one contangent vector, and three tangent vectors) and is linear since the RHS is linear in each argument by definition, so T is a tensor.

———————————————————————

iv)$T^D_{ABC} = {H^D}_A + M_{BC}$

****

This is not a tensor. It is not a linear function of the arguments.

$$(T^D_{ABC} X^A Y^B Z^C (V_D + U_D) = {H^D}_A (V_D + U_D) + M_{BC} Y^B Z^C \tag{30}$$
$$= {H^D}_A V_D + {H^D}_A U_D + M_{BC} Y^B Z^C \tag{31}$$

while

$$T^{D}_{ABC}X^{A}Y^{B}Z^{C}V_{D} + T^{D}_{ABC}X^{A}Y^{B}Z^{C}U_{D} = \tag{32}$$
$$H^{D}{}_{A}V_{D} + M_{BC}Y^{B}Z^{C} + H^{D}{}_{A}U_{D} + M_{BC}Y^{B}Z^{C} \tag{33}$$

which is not the same. (there are two $M_{BC}Y^{B}Z^{C}$)

———————————————————————

v) $R^{A} = L_{B}{}^{A}{}_{B}$

****

This is not a tensor. There is not operation which "contracts" indices of the same type. this could only mean that the third argument of L is also the first, but then R is a function of only one argument. So the arguments on the two sides are different so they could not be the same as functions.

———————————————————————

vi) $S_{A} = L_{A}{}^{B}{}_{B} - L_{B}{}^{B}{}_{A}$

****

This time, the equal index on the RHS does have a meaning, that of contraction, which is a valid tensor operation.

———————————————————————

vii) $T_{A} = \nabla_{B}H^{B}{}_{A}$

****

$\nabla_{C}H^{B}{}_{A}$ would be a tensor of three arguments, with the first one and second one being tangent and cotangent vectors respectively. Thus contraction of these two arguments is defined, and is a tensor operation.

———————————————————————

Is $\partial_{i}H^{j}{}_{k}$ the component of a tensor?

****

No, this is not. This only a part of the covariant derivative, the remaining part being the covariant derivative of the coordinate basis vectors (expressed in terms of the Christofel symbols). The coordinates of the covariant derivative would be

$$\nabla_{i}H^{j}{}_{k} = \partial_{i}H^{j}{}_{k} + \sum_{m}(\Gamma^{j}_{mi}H^{m}{}_{k} - \Gamma^{m}_{ki}H^{j}{}_{m} \tag{34}$$

———————————————————————-

What are the components of expressed in terms of partial derivatives, Christofel symbols?

$$\nabla_{A}H^{A}{}_{B} \tag{35}$$

4. Given coordinates $r, theta$, what are the tangent vectors to the curves defined by the coordinate conditions expressed in terms of $\frac{\partial}{\partial r}^{A}$ and $\frac{\partial}{\partial \theta}^{A}$

$$r(\lambda) = r_{0} \tag{36}$$
$$\theta(\lambda) = \lambda \tag{37}$$

$$r(\lambda) = \lambda \tag{38}$$
$$\theta(\lambda) = 5 * \lambda \tag{39}$$

$$r(\lambda) = 10\lambda \tag{40}$$
$$\theta(\lambda) = 50 * \lambda \tag{41}$$

**8

$$\frac{\partial}{\partial\gamma}^{A} = \frac{\partial}{\partial\theta}^{A} \tag{42}$$

$$\frac{\partial}{\partial\gamma}^{A} = \frac{\partial}{\partial r}^{A} + 5\frac{\partial}{\partial\theta}^{A} \tag{43}$$

$$\frac{\partial}{\partial\gamma}^{A} = 10\frac{\partial}{\partial r}^{A} + 50\frac{\partial}{\partial\theta}^{A} \tag{44}$$

———————————————————————-

What is the cotangent vector of the following functions

$$f(r,\theta) = r^2 \tag{45}$$

$$f(r, theta) = r^2 + \theta^2 \tag{46}$$

****

$$df_A = 2r dr_A \tag{47}$$

$$df_A = 2r dr_A + 2\theta d\theta_A$$

——————————————————

In each case find the lengths of these various vectors for each point at which they are defined if the metric is given by a)

$$ds^2 = dr^2 + d\theta^2 \tag{48}$$

****

The inverse metric is the same matrix as the metric.. Thus the lengths are 1

$$\sqrt{26}$$

$$10\sqrt{26}$$

2r

$$2\sqrt{r^2+\theta^2}$$

————————————————————————

and

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{49}$$

****

1

$$\sqrt{1+5^2r^2} = \lambda\sqrt{26}$$

$$\sqrt{100+50^2r^2} = 10\lambda\sqrt{26}$$

2r

$$2\sqrt{r^2+\frac{1}{r^2}\theta^2}$$

—————————————————————

5) What are all the components of the Christofel symbols for the metric

$$ds^2 = \frac{1}{r}dr^2 + r d\theta^2 \tag{50}$$

and for

$$ds^2 = \frac{1}{r}dr^2 - r dt^2 \tag{51}$$