Physics 407-08

Scwartzschild metric geodesics

In the early months of 1917, while an artillery officer on the Eastern front, Schwartzschild found the first exact solution of Einstein's field equations, published a month earlier.

The solution he found had the metric form

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{1}$$

This is a symmetric metric. It does not depend on time, so it has a time translation symmetry. The only dependence on the two angles θ ϕ is in the metric of a uniform sphere $d\theta^2 + sin(\theta)^2 d\phi^2)$. Since the distance around the circumpherence of the sphere, t and r both constant and $\theta = \pi/2$ is $2\pi r$, the coordinate r has been chosen to be given by the circumference of the spheres.

This metric has four independent Killing vectors,

$$K^i_{(t)} = [1, 0, 0, 0] \tag{2}$$

$$K^i_{(z)} = [0, 0, 0, 1] \tag{3}$$

$$K^i_{(x)} = [0, 0, cos(\phi), -sin(\phi)cot(\theta)] \tag{4}$$

$$K^i_{(x)} = [0, 0, -sin(\phi), cos(\phi)cot(\theta)] \tag{5}$$

The first two are obvious because nothing in the metric depends on either ϕ or t. The last two are not so obvious but can be shown to obey the Killing equation

$$\sum_k (K^k \partial_k g_{ij} + g_{ik}\partial_j K^k + g_{kj}\partial_i K^k) = 0 \tag{6}$$

The last three Killing vectors correspond to the three rotations (about the z,x y axes) that one would expect for a spherically symmetric metric.

Since another way of writing the Killing equation is in terms of the parallel derivative

$$D_A K_B + D_B K_A = 0 \tag{7}$$

For a geodesic (straight line) we find

$$D_s(K_A U^A) = (D_s K_A)U^A + K_A D_s U^A = U^B D_B K_A U_A + 0 = 0 \tag{8}$$

The first 0 is because U^A is the unit tangent vector to the curve and for a geodesic, the unit tangent vector has parallel derivative of 0. The second 0 is because $D_B K_A U^B U^A = D_A K_B U^B U^A$.

Thus the contraction of the 4 Killing vectors with U^A are all constants of motion. The three angular Killing vectors are can all be rotated so that the two of the constants are 0. We choose $K_{(z)A} U^A = L$ as the non-zero one. The other two being zero imply that U^θ is zero and that $cot(\theta)$ is zero– ie the orbit lies in the equatorial plane.

We thus have

$$g_{tt} U^t = E \tag{9}$$

$$g_{\phi\phi} U^\phi = L \tag{10}$$

Or

$$\frac{dt}{ds} = \frac{E}{1 - \frac{2M}{r}} \tag{11}$$

$$\frac{d\phi}{ds} = \frac{L}{r^2} \tag{12}$$

(where I have used the fact that $sin(\phi) = 1$.) Using the normalisation of the tangent vector

$$\frac{1}{1 - \frac{2M}{r}} (\frac{dr}{ds})^2 - \frac{E^2}{1 - \frac{2M}{r}} + \frac{L^2}{r^2} = -1 \tag{13}$$

If we are interested in the shape of the orbit, we can rewrite this equation in terms of ϕ

$$\frac{dr}{ds} = \frac{dr}{d\phi}\frac{d\phi}{ds} = \frac{dr}{d\phi}\frac{L}{r^2} = -\frac{d\frac{1}{r}}{d\phi} L$$

Defining $u = \frac{1}{r}$ we get

$$L^2 (\frac{du}{d\phi})^2 - E^2 + (1 - 2Mu)(u^2) = -1 \tag{14}$$

or

$$(\frac{du}{d\phi})^2 = \frac{E^2 - (1 - 2Mu)}{L^2} - (1 - 2Mu) u^2 \tag{15}$$

The right hand side is a cubic equation, which can be written as

$$(\frac{du}{d\phi})^2 = 2M(u_0 - u)(u1 - u)(u - u2) \tag{16}$$

where

$$u_0 + u_1 + u_2 = \frac{1}{2M} \tag{17}$$
$$u_0u_1 + u_0u_2 + u_1u_2 = \frac{1}{L^2} \tag{18}$$
$$u_0u_1u_2 = \frac{E^2 - 1}{2ML^2} \tag{19}$$

If I define $\bar{u} = (u_1 + u_2)/2$ and $\Delta = (u_1 - u_2)/2$ (**Note that this Δ is 1/2 of the definition I used in the lectures**), and we assume that L is very large, and that $E^2 < 1$ (ie the orbit is a bound orbit), we have that $\bar{u}$ is very small. If we assume that the orbit is nearly circular, we have the $\Delta << \bar{u}$. We can write

$$\int d\phi = \int \frac{du}{\sqrt{2M(u_0 - u)(u_1 - u)(u_2 - u)}} \tag{20}$$
$$= \int \frac{du}{\sqrt{(1 - 6M\bar{u} - (u - \bar{u}))(\Delta^2 - (u - \bar{u})^2)}} \tag{21}$$

Thus u must lie between $\bar{u} \pm \Delta$ and by assumption since Δ is very small, u is very close to $\bar{u}$.Thus we can write

$$\int \frac{du}{\sqrt{(1 - 6M\bar{u} - (u - \bar{u}))(\Delta^2 - (u - \bar{u})^2)}} \tag{22}$$
$$\approx \int \frac{du}{\sqrt{(1 - 6M\bar{u})(\Delta^2 - (u - \bar{u})^2)}} \tag{23}$$
$$= \frac{1}{\sqrt{(1 - 6M\bar{u})}} arccos(\frac{u - \bar{u}}{\Delta}) \tag{24}$$

or

$$u = \bar{u} + \frac{\Delta}{2} cos(\sqrt{(1 - 6M\bar{u})}\phi \tag{25}$$

Thus the orbit will return to the initial value of u after an angle of

$$\sqrt{(1-6M\bar{u})}\phi = 2\pi \tag{26}$$

or

$$\delta\phi = \frac{2\pi}{\sqrt{(1-6M\bar{u})}} \approx 2\pi(1+3M\bar{u}) \tag{27}$$

Thus in one orbit, the angle at which closest approach(largest u) occurs advances by $6\pi M\bar{u} = \frac{6\pi M}{R}$ where R is the radius of the orbit.

This prediction that in General Relativity the perihelion will advance was one of the first indications that General Relativity was right. In the 19th century, the calculation of Mercury's orbit showed that there was an unexplained advance of the perihelion of about 42 seconds of arc per century. The prediction of General Relativity was just 42 seconds of arc per century.

0.1 Bending of light

The only difference between the orbit of light or of a massive particle is that the length of the tangent vector for light is 0. This implies that the equation for u as a function of ϕ is

$$(\frac{du}{d\phi})^2 = \frac{E^2}{L^2} - u^2(1-2Mu) \tag{28}$$

Again the right hand side can be written as cubic with the three roots u_0 u_1 u_2 in order of the biggest to smallest roots. In this case

$$u_0 = \frac{1}{2M} - 2\bar{u} \tag{29}$$

$$u_0u_1 + u_0u_2 + u_1u_2 = 0 \tag{30}$$

$$2Mu_0u_1u_2 = -\frac{E^2}{L^2} \tag{31}$$

or

$$\frac{1}{2M}\bar{u} - 2\bar{u}^2 + (\bar{u}^2 - \Delta^2) = 0 \tag{32}$$

Assuming that $\bar{u}$ and Δ are very small, we have

$$\bar{u} \approx 2M\Delta^2 \tag{33}$$

Ie, $\bar{u}$ is much smaller than is Δ. We can again integrate the equation. Assuming that ϕ increases as r increases (or as u decreases) we have

$$\begin{aligned}
\phi &= -\int \frac{du}{\sqrt{(1-6M\bar{u}-2M(u-\bar{u}))(\Delta^2-(u-\bar{u})^2)}} && (34)\\
&= -(1+3M\bar{u})acos(\frac{u-\bar{u}}{\Delta}) - \int \frac{M(u-\bar{u})du}{(\Delta^2-(u-\bar{u})^2)} && (35)\\
&= (1+3M\bar{u})acos(\frac{u-\bar{u}}{\Delta}) + M\Delta\sqrt{1-\frac{(u-\bar{u})^2}{\Delta^2}} && (36)\\
&\approx acos(\frac{u-\bar{u}}{\Delta}) + M\Delta\sqrt{1-\frac{(u-\bar{u})^2}{\Delta^2}} && (37)
\end{aligned}$$

Thus the deflection angle from $u = \bar{u} + \Delta$ (the closest approach) to $u = 0$ (which is $r = \infty$) is

$$\delta\phi = acos(-\frac{\bar{u}}{\Delta}) + M\Delta = \frac{\pi}{2}\frac{\bar{u}}{\Delta}) + M\Delta = \frac{\pi}{2} + 2M\Delta \tag{38}$$

Since to lowest order $r_{min} = 1/(\Delta + \bar{u}) \approx 1/\Delta$, we have that the total deflection angle is twice the deflection from the closest approach to infinity, which is

$$2\delta\phi = \pi + 4\frac{M}{r_{min}} \tag{39}$$

For the sun at the limb of the sun, this is approximately 1.75 seconds of arc. (that is 48 microns in a 10 metre telescope– the accuracy to which Dyson and Eddington had to measure the displacement of the stars on their photographic plates if the star image were right next to the limb of the sun.)

Note that Hipparcos sattelite measures the location of stars to of order a milli arc-second. Thus they would be able to see the deflection of starlight over 1000 times the radius of the sun. Since the earth's orbit is only about 100 times the radius of the sun, they could easily measure the deflection of stars whose light had an impact parameter of earth's orbit. Ie, one of the larger corrections they had to make to the positions of the stars was that due to the deflection of the light by the sun. In fact, since the earth's mass is about 10^{-6} of the sun's and the radius of the earth is about 1/100th that of the sun, the deflection of light by the earth is almost at the limit of detectability by Hipparcos, and will be by the next astrometry sattelite, Gaia whose accuracy will be 25 micro arc-seconds.

0.2 Time delay

In 1964 I. Shapiro suggested another test of the General Relativity, the delay of a light beam passing near the sun or other gravitating body. In the late 70s he measured it by observing the passage of radar signals past the sun by sending a pulse from the earth to venus and timing the return of the signal (as can be imagined the returning signal was rather weak.)

We look at the equation of motion for light as metioned above. We know that its path is, to first order in M given by

$$u - \bar{u} = \Delta cos(\phi - M\Delta\sin(\phi)) \approx \Delta cos(\phi) + M\Delta^2 sin^2(\phi) \tag{40}$$

The time is given by

$$\frac{dt}{ds} = \frac{E}{1-2Mu} \tag{41}$$

or, expressing this in terms of u

$$\frac{dt}{du}\frac{du}{ds} = -\frac{E}{1-2Mu} \tag{42}$$

$$\frac{dt}{du} = \frac{E}{L}\frac{1}{u^2(1-2Mu)\sqrt{2M(u_0-u)(u_1-u)(u-u_2)}} \tag{43}$$

Thus we have

$$t = \int \Delta \frac{1+3Mu}{u^2\sqrt{\Delta^2-(u-\bar{u})^2}}du \tag{44}$$

$$\approx \frac{\sqrt{1-\frac{(u-\bar{u})^2}{\Delta^2}}}{u} + 2M\ln\left(\frac{1+Mu+\sqrt{1-\frac{(u-\bar{u})^2}{\Delta^2}}}{u}\right) \tag{45}$$

letting $u = 1/r$ and $\Delta = 1/r_{min}$ we get

$$t = \sqrt{r^2 - r_{min}^2} + 2M \ln(r + \sqrt{r^2 - r_{min}^2}) \tag{46}$$

For large r, the first term is just what we would expect from Newtonian theory (if $\sqrt{r^2 - r_{min}^2}$ is the disntance travelled) and the second term blows up logarithmically in r. It is the what is usually called the Shapiro time delay. For large r the delay on going from r_{min} to r is

$$\delta t \approx 2M \ln(\frac{2r}{r_{min}}) \tag{47}$$

If it comes from a source, the same is true of the source. Ie, the total delay is

$$2M \ln(\frac{4 r_s r_o}{r_{min}^2}) \tag{48}$$

where r_s is the distance the source is away from the deflecting body, while r_o is the distance the observer is away.