Preprint typeset in JHEP style - HYPER VERSION

Cosmology
Lectures given to Physics 407 Sep 27-Oct3 08

Silke Weinfurtner

University of British Columbia, silke.weinfurtner@ubc.ca and unruh@physics.ubc.ca

Abstract: These lecture notes are an informal and ad hoc introduction to cosmology. They are part of an introductory course for general relativity within the UBC undergraduate program. While we do expect the reader to be familiar with the notion of spacetime, an understanding of the Einstein equations is not required. The lectures are organized as follows: To begin with we summarize key features of the universe found through observations, to then build a theoretical model reproducing the universe as a whole. Before we start we would like to point out that these notes are influenced by James Hartle's book "Gravity: An Introduction to Einstein's General Relativity".

Keywords: Cosmology.

Contents

Cosmology is the study of our universe as a whole. That is the structure and evolution of the universe on the largest scales of space and time. On these so called cosmological scales we are able to apply general relativity to the entire universe. Is there one spacetime geometry, g_{ab}, capable of describing the universe as a whole? The answer is yes, and in the following we first discuss observational facts and then construct a theoretical model to match our universe.

1. Observations of the universe

Let us look at the sky with the Hubble deep field (HdF) teleskope, see Figure 1, and raise the question "what is really out there"? The answer is that we are living in a universe

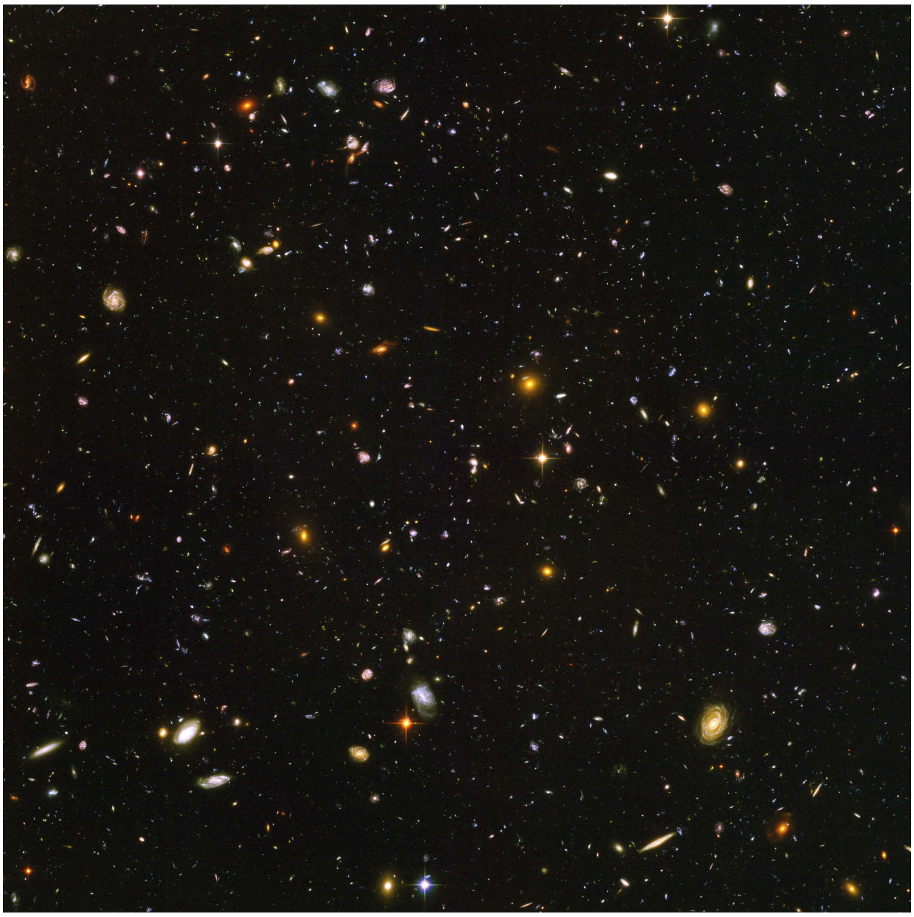


Figure 1: Universe full of galaxies: The visible universe contains about 80 billion galaxies.

full of galaxies. A similar, yet more detailed, presentation of the subject can be found in James Hartle's book [1], while we refer to Scott Dodelson's book on "Cosmology", see [2], as advanced reading material.

1.1 Composition of the universe (O1)

Galaxies are gravitationally bound collections of stars, gas and dust. Let us study in detail what the galaxies are made off and shortly we will see that almost all of the energy content in our universe is invisible (dark).

1.1.1 Matter

One galaxy has approximately 10^{10} stars and its total mass is about 10^{12} sun masses. The visible energy density is given as

$$\rho_{\text{visible}} \sim 10^{-31}\, g/cm^3. \tag{1.1}$$

1.1.2 Radiation

Radiation summarizes zero rest mass particles, e.g., photons, neutrinos (almost zero mass), and gravitational waves. Radiation does not cluster gravitationally. Radiation with a relatively high energy density,

$$\rho_r(t_0) \sim 10^{-34}\, g/cm^3, \tag{1.2}$$

has been detected in the cosmological microwave background (CMB). The CMB radiation is electromagnetic radiation left over from the big bang and a perfect example for a blackbody spectrum, , see Figure 2. The temperature measured today is $T_{\text{CMB}} \sim 2.725 \pm 0.001\,\text{K}$. In this sense the CMB is one of the strongest evidences for the big bang scenario.

1.1.3 Dark matter

Besides visible matter there is very strong evidence that the universe contains a very large portion of dark matter. Dark matter is supposed to be matter that does not interact with the electromagnetic force. It only interacts gravitationally. Therefore its existence can be inferred from gravitational effects on visible matter. For example, observations of rotation curves in spiral galaxies cannot be explained with the visible mass-distribution. The angular velocities are supposed to scale with

$$|F_g| = |F_c| \tag{1.3}$$

$$\frac{G\,M(r)}{r^2} = \frac{V^2(r)}{r} \quad \rightarrow \quad V(r) \sim \frac{1}{r}. \tag{1.4}$$

Rotation curves observed in the nearby spiral galaxy M33, see Figure 3, do not match the predicted velocity profile: The observed velocity distribution (solid line) cannot be explained with the visible mass (short dashed line) and radiation distribution (long dahed line). Concluding that there must be some sort of dark matter extending the mass distribution outside the visible core of the galaxy (short-long dashed line).

There was quite an interest in the subject and during the class and I got many good questions. I would like to answer one particular question and add them to the lecture notes:

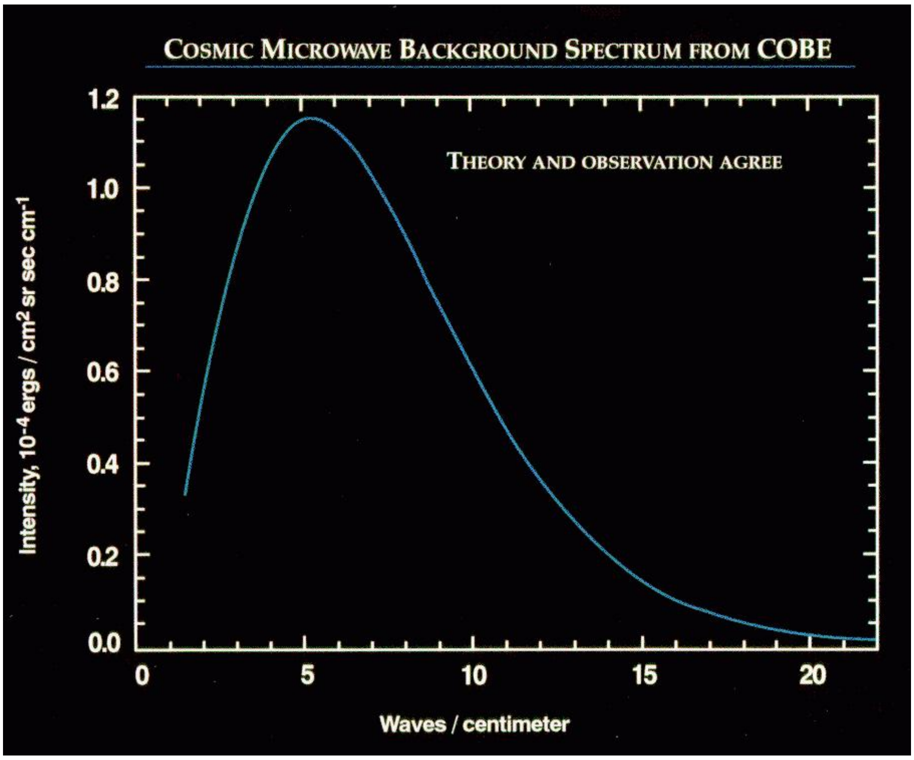


Figure 2: The cosmological microwave background is a perfect example for a blackbody spectrum. The peak of the spectrum lies in the microwave range.

Why is it that dark matter seems to form a halo around the core of the visible matter? The answer is that it does not, it forms a thermally buoyed dust ball. Further we would like to mention that dark matter matter will not form a nucleus surrounded by non-dark matter as it has no mechanism for radiative cooling and thus it is easier for visible baryons to collapse and form galaxies.

1.1.4 Dark energy

Dark energy shares one essential feature with dark matter, it only interacts gravitationally, but unlike dark matter it does not like to clump. Dark energy has zero rest mass and negative pressure and is supposed to explain the expansion of the universe.

Again, there was quite an interest in the subject and I thought that it might be worth to add a more detailed description to the notes. I took the following definition from wikipedia:

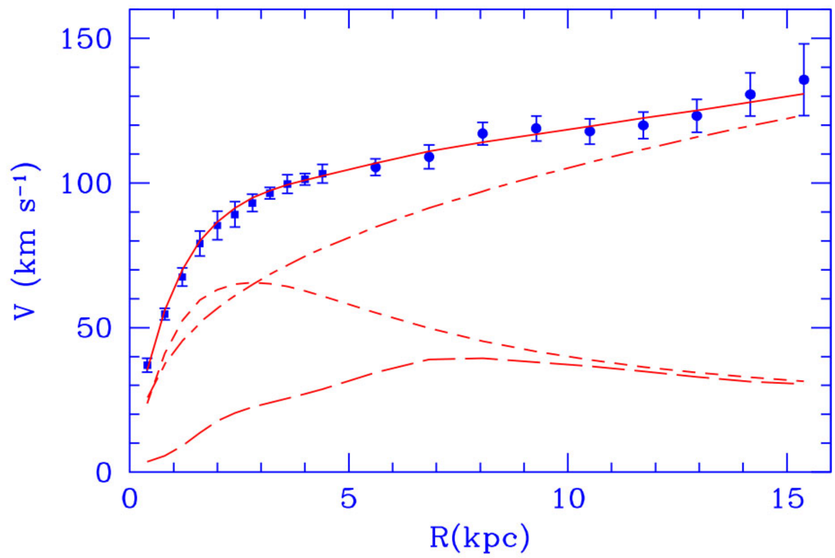


Figure 3: Rotation curve for stars in the nearby galaxy M33.

Wikipedia on dark energy: "In physical cosmology, dark energy is a hypothetical exotic form of energy that permeates all of space and tends to increase the rate of expansion of the universe. Dark energy is the most popular way to explain recent observations that the universe appears to be expanding at an accelerating rate. In the standard model of cosmology, dark energy currently accounts for 74% of the total mass-energy of the universe. Two proposed forms for dark energy are the cosmological constant, a constant energy density filling space homogeneously, and scalar fields such as quintessence or moduli, dynamic quantities whose energy density can vary in time and space. Contributions from scalar fields that are constant in space are usually also included in the cosmological constant. The cosmological constant is physically equivalent to vacuum energy. Scalar fields which do change in space can be difficult to distinguish from a cosmological constant because the change may be extremely slow."

1.1.5 Summary

Our universe contains visible and dark matter, radiation and dark energy and it is believed to split as illustrated in Figure 4.

1.2 The expanding universe (O2)

Up to now we were only concerned about the the content of the universe, but we do not know if and how it evolves in time. Spectra of starlight from galaxies outside our local group are redshifted, implying that everything is moving away from each other. The redshift parameter is z and defined as the ratio between the velocity of the receding galaxy and the speed of light, equivalent to the shift in the wavelength to its value as measured in our restframe,

$$\frac{v}{c} = \frac{\Delta\lambda}{\lambda} = z\,. \tag{1.5}$$

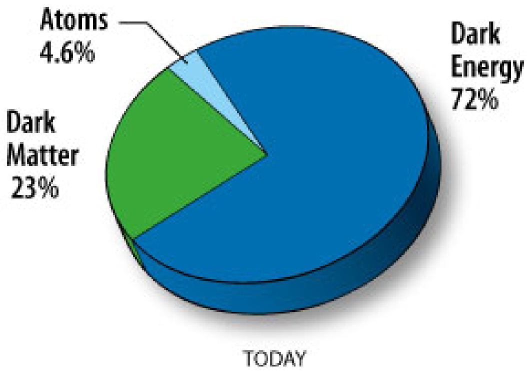


Figure 4: The chart shows the currently believed energy density distribution.

In Figure 5 it is illustrated how the four spectral lines of hydrogen are redshifted due to the expansion of our universe. Common atomic sources for spectral lines, besides hydrogen, that are relevant for redshift observations are sodium, magnesium, calcium and iron, see Figure 6.

1.2.1 Hubble's law

Hubble's law is a phenomenological relationship between the redshift and distance from galaxies

$$V = H_0(t)\, d\,. \tag{1.6}$$

It is applicable for galaxies that are sufficiently far away such that gravitational attraction of nearby galaxies are neglectable. However it should not be too far away such that the expansion of the universe does not significantly effects light rays. Even though $H_0(t)$ can be time-dependent it is called the Hubble constant. This is due to the fact that we can only measure the Hubble constant at an instant of time.

The Hubble law has been tested observationally by measuring both the distance and (indirectly) the velocity of receding galaxies (through spectral redshift, see equation (1.5),

$$H_0\, d = c\, z \quad \rightarrow \quad H_0 = \frac{c\, z}{d}\,. \tag{1.7}$$

The current value from observations is

$$H_0 = 72 \pm 7\ (\mathrm{km/s})/\mathrm{Mpc}. \tag{1.8}$$

Up to now we have not discussed the techniques used to measure distances of far away galaxies. Here we will focus on one particular experiment involving type Ia supernovae (SNIa). Supernovae events refer to the collapse of a massive star that has exhausted all its thermonuclear fuel. For the case of a type Ia supernovae is a binary star system between a normal star and a white dwarf (a small compact star mostly consisted of electron-degenerate matter). Due to the mass transfer from the normal star to the white dwarf the mass of the white dwarf eventually reaches the maximum mass at which the pressure of the degenerate

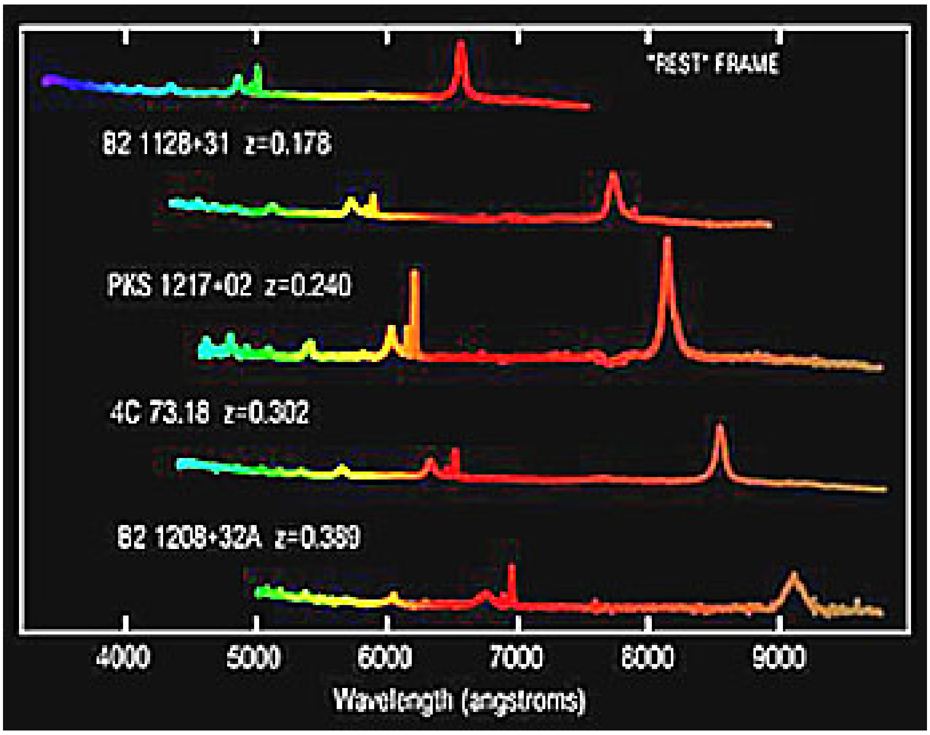


Figure 5: Redshift of starlight from a distant galaxy causes all band to be shifted about the same amount. The amount about which the wavelengths are shifted depends on the distance of the galaxy. Here we look at five spectra from the Kitt Peak National Observatory telescope. In each spectra we can observe three different lines for the hydrogen atom. At the rest frame we can see that the lines are at 4340 A (in the blue), 4860 A (green), and 6552 A (red). The redshift increases from the top down.

electrons causing the star to erupt in a powerful fuel nuclear burning. The luminosity of this event is known and since the apparent brightness (flux on earth) depends on its distance,

$$f = \frac{L}{2\pi d^2}, \tag{1.9}$$

we can use this so-called standard candles to measure d. Figure 7 shows the data obtained from a SNIa.

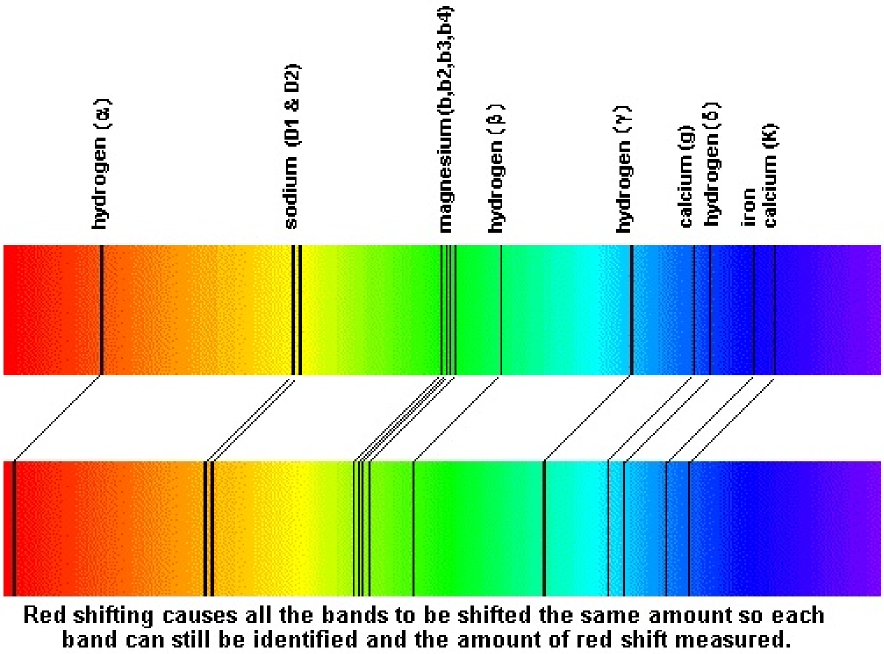


Figure 6: Spectra lines of common atomic elements.

1.3 Isotropy and Homogeneity (O3)

So far we have discussed the content of our universe and the fact that all galaxies are receding from each other. We are living in an expanding universe. However the question remains how this universe is organized on large scales. Are all observers equal, or could it be that we are accidentally live in a very dense neighbourhood? In other words is our universe homogeneously distributed in space? And similarly important is the expansion isotropic, or does the universe favour a specific direction? Below we will show that we have strong observational evidence that the universe is isotropic and homogeneous.

1.3.1 Isotropy

How does our universe now, that is about 13 billion years after the big bang, looks like on large scale? In section 1.1.2 we looked the electromagnetic radiation left over from the big bang. The CMA is also an excellent source to study isotropy of our universe. In Figure 8 we see data taken from the COBE satellite (NASA's Goddard Space Flight Center) developed to measure the diffuse infrared and microwave radiation from the early universe. Maps based on the 53, that is 5.7 mm wavelength, observations show (see top picture in Figure 8) that the universe is isotropic well above the millikelvin scale. At higher resolution, at the millikelvin scale, we discover dipole anisotropy due to the motion of our solar system with respect the rest frame of the background radiation (middle picture). After subtraction of the dipole component we are left with anisotrophy responsible for structure formation in our early universe.

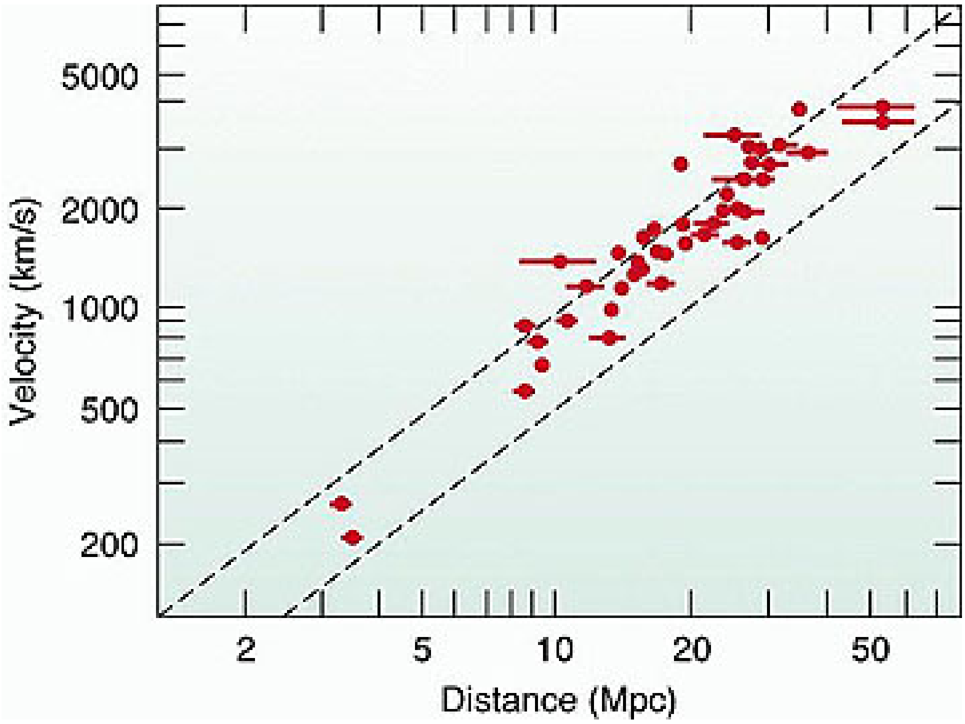


Figure 7: Detection of the Hubble constant H_0 from type Ia supernovae data.

1.3.2 Homogeneity

Three dimensional maps of our spatial universe show that the galaxy distribution in our universe is homogeneous. Thus has no large scale formation at cosmological scales. There are two observations confirming this assumption, the 2dF, Figure 9(a), and sloan digital sky survey, Figure 9.

2. A theoretical model for our observations

Our universe is composed of (visible and dark) matter, radiation and dark energy (possibly related to the cosmological constant, a constant homogenous energy density), is expanding and homogenous and isotropic at cosmological scales. In the following we will try to construct a theoretical model that all of this features and thus matches the observations discussed earlier on.

2.1 Kinematics - Friedman–Robertson–Walker spacetimes (T1)

We assume that on cosmic scales it is possible to describe the evolution of the universe as a whole. At that scales gravity is the dominant theory. According to general relativity gravity is not a force but a consequence spacetime geometry. In the following we make use of our observations to derive a geometrical interpretation for our universe.

It can be shown that for a spherically symmetric homogeneous spacetime, $(-,+,+,+)$, that is a maximally symmetric three-dimensional subspace with positive eigenvalues and

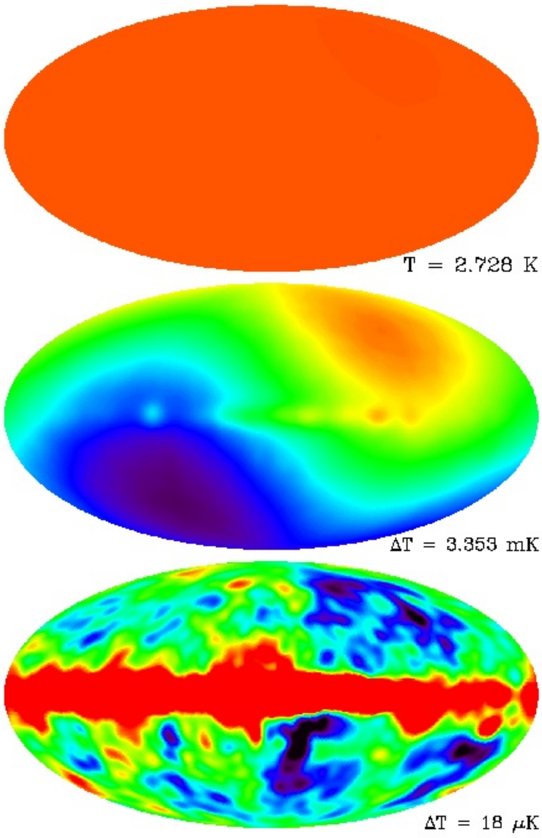


Figure 8: CMB map taken by the COBE telescope.

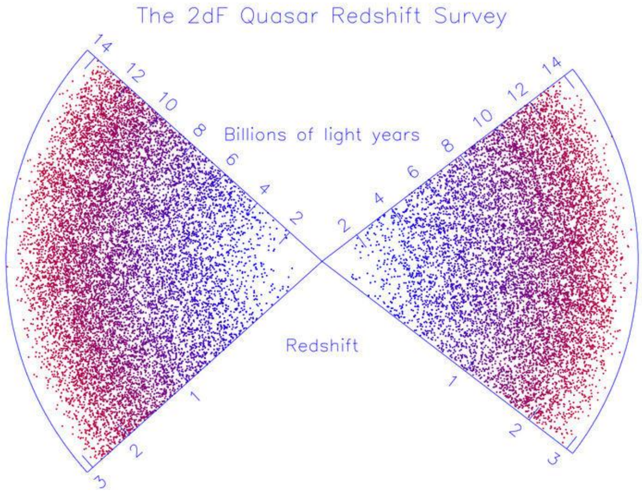
arbitrary curvature, the line element is given by

$$ds^2 = -dt^2 + a(t)^2 \, d\mathcal{L}^2. \tag{2.1}$$

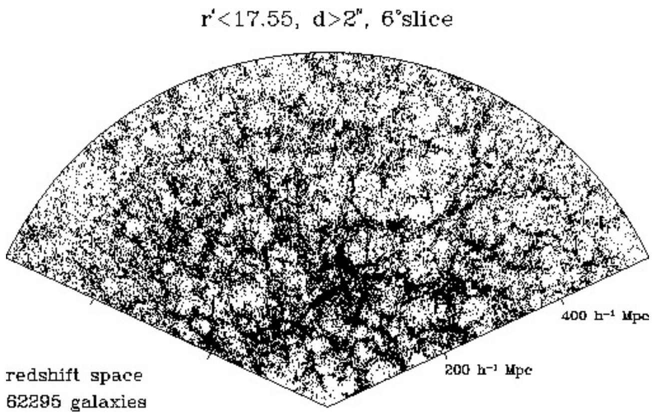
Here $a(t)$ is the scale factor of our universe and $d\mathcal{L}^2$ is a time-independent homogeneous isotropic three-dimensional space $(+,+,+)$.

Note that this line element can be derived by symmetry arguments, more specifically symmetry of subspaces, and it is not necessary to involve any dynamics, here the Einstein equations, to uniquely pin down the geometrical spacetime as stated in equation (2.1). A rigours derivation can be found in Weinberg's book on "Gravitation and Cosmology" [3]. [1]

[1]The book can be downloaded from http://www.scribd.com/doc/3207044/Weinberg-Gravitation-and-Cosmology-Principles-and-Applications-of-the-General-Theory-of-Rel.



(a) The 2dF survey.



(b) The sloan digital sky survey.

Figure 9: Observational evidence for a homogenious universe.

2.1.1 What homogeneous isotropic three-dimensional spaces are there?

There are three different, with positive, negative and zero curvature, spaces and they are represented as follows:

- Euclidean space in $\mathbb{R}^3$ given by $d\mathcal{L}^2 = dx^2 + dy^2 + dz^2$.
- Two embedings in $\mathbb{R}^4$ that have a homogeneous isotropic three-dimensional subspace S^3.

Before we study the two embedding in detail we first look at the lower-dimensional cases $S^2 \in \mathbb{R}^3$, see examples as shown in Figure 10. There we argue that only two of the embedding are suitable candidates, that are (a) and (b). In the following we extend this idea for $\mathcal{S}^3 \in \mathbb{R}^4$, such that we obtain purely spatial hypersurfaces representing the three different spatial curvatures. That are spherical (sphere embedded in $\mathbb{R}^4$), flat ($\mathbb{R}^3$) and

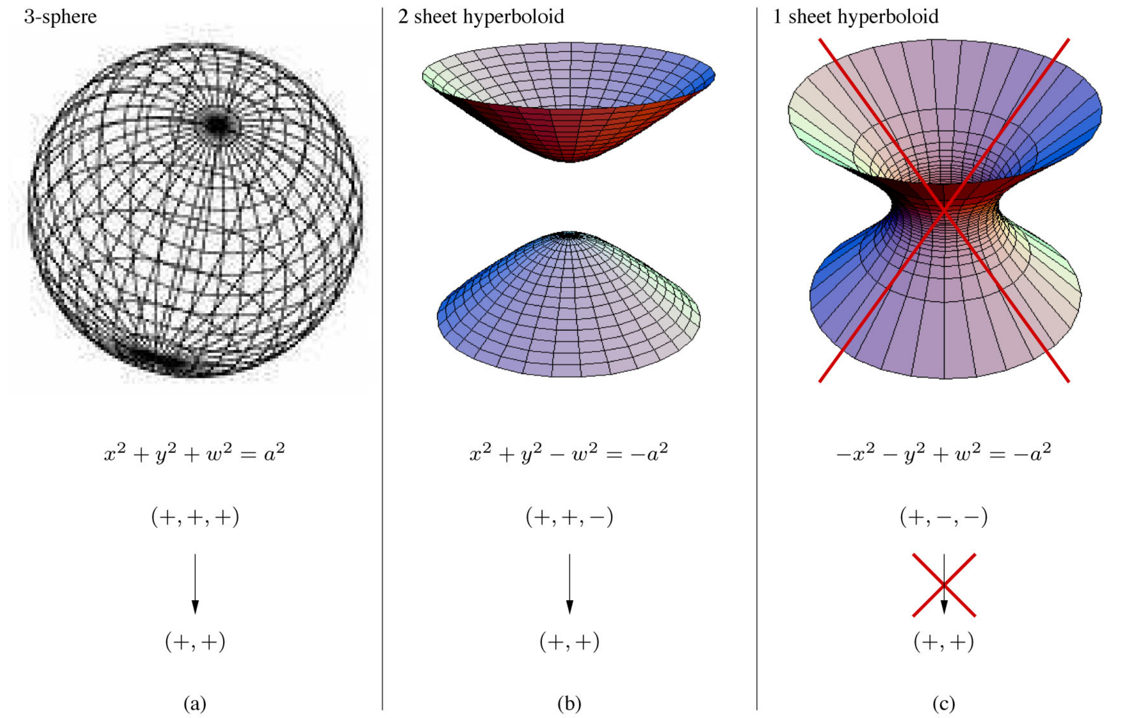


Figure 10: Examples for two-dimensional embedding in $\mathbb{R}^3$. Here we see three different embedding in $\mathbb{R}^3$. Only (a) and (b) have purely two-dimensional spacial surfaces.

hyperbolical (upper half of 1 sheet hyperboloid embedded in $\mathbb{R}^4$) spaces,

$$d\mathcal{L}^2 = \begin{cases} dx^2 + dy^2 + dz^2 + \frac{(x\,dx + y\,dy + z\,dz)^2}{a^2 - x^2 - y^2 - z^2} & \text{where} \quad x^2 + y^2 + z^2 + w^2 = a^2 \\ dx^2 + dy^2 + dz^2 & \\ dx^2 + dy^2 + dz^2 + \frac{(x\,dx + y\,dy + z\,dz)^2}{a^2 + x^2 + y^2 + z^2} & \text{where} \quad x^2 + y^2 + z^2 - w^2 = -a^2 \end{cases} . \tag{2.2}$$

Here a is a constant, w the fourth dimension which has been removed using constraint equations, see Figure 11. After appropriate coordinate transformations for the sphere

$$x = a\,\cos\phi\,\sin\theta\,\sin\chi \tag{2.3}$$
$$y = a\,\sin\phi\,\sin\theta\,\sin\chi \tag{2.4}$$
$$z = a\,\cos\theta\,\sin\chi \tag{2.5}$$
$$w = a\,\cos\chi\,, \tag{2.6}$$

for the plain

$$x = \chi\,\cos\phi\,\sin\theta \tag{2.7}$$
$$y = \chi\,\sin\phi\,\sin\theta \tag{2.8}$$
$$z = \chi\,\cos\theta\,, \tag{2.9}$$

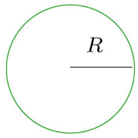
space	$d\mathcal{L}^2$	
flat	$d\mathcal{L}^2 = dx^2 + dy^2 + dz^2$	$u = 2\pi\, R$
sphere	$w^2 = a^2 - x^2 - y^2 - z^2$ $dw^2 = \frac{(-xdx-ydy-zdz)^2}{a^2-x^2-y^2-z^2}$ $d\mathcal{L}^2 = dx^2 + dy^2 + dz^2 + \frac{(xdx+ydy+zdz)^2}{a^2-x^2-y^2-z^2}$	$u < 2\pi\, R$
hyperboloid	$w^2 = a^2 + x^2 + y^2 + z^2$ $dw^2 = \frac{(xdx+ydy+zdz)^2}{a^2+x^2+y^2+z^2}$ $d\mathcal{L}^2 = dx^2 + dy^2 + dz^2 + \frac{(xdx+ydy+zdz)^2}{a^2+x^2+y^2+z^2}$	$u > 2\pi\, R$

Figure 11: Examples for two-dimensional embedding in $\mathbb{R}^3$.

and for the hyperboloid

$$x = a\, \cos\phi\, \sin\theta\, \sinh\chi \tag{2.10}$$
$$y = a\, \sin\phi\, \sin\theta\, \sinh\chi \tag{2.11}$$
$$z = a\, \cos\theta\, \sinh\chi \tag{2.12}$$
$$w = a\, \cosh\chi\,, \tag{2.13}$$

see also Figure 12, we get altogether

$$ds^2 = -dt^2 + a(t)^2 \left[d\chi^2 + \begin{Bmatrix} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{Bmatrix} \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right]. \tag{2.14}$$

We could stop right here, but for further connivence we perform yet another coordinate transformation,

$$r = \sin\chi \quad \rightarrow \quad \chi = \arcsin r \quad \rightarrow \quad d\chi = \frac{1}{\sqrt{1-r^2}}\, dr \tag{2.15}$$
$$r = \chi \quad \rightarrow \quad d\chi = dr \tag{2.16}$$
$$r = \sinh\chi \quad \rightarrow \quad \chi = \operatorname{arcsinh} r \quad \rightarrow \quad d\chi = \frac{1}{\sqrt{1+r^2}}\, dr\,. \tag{2.17}$$

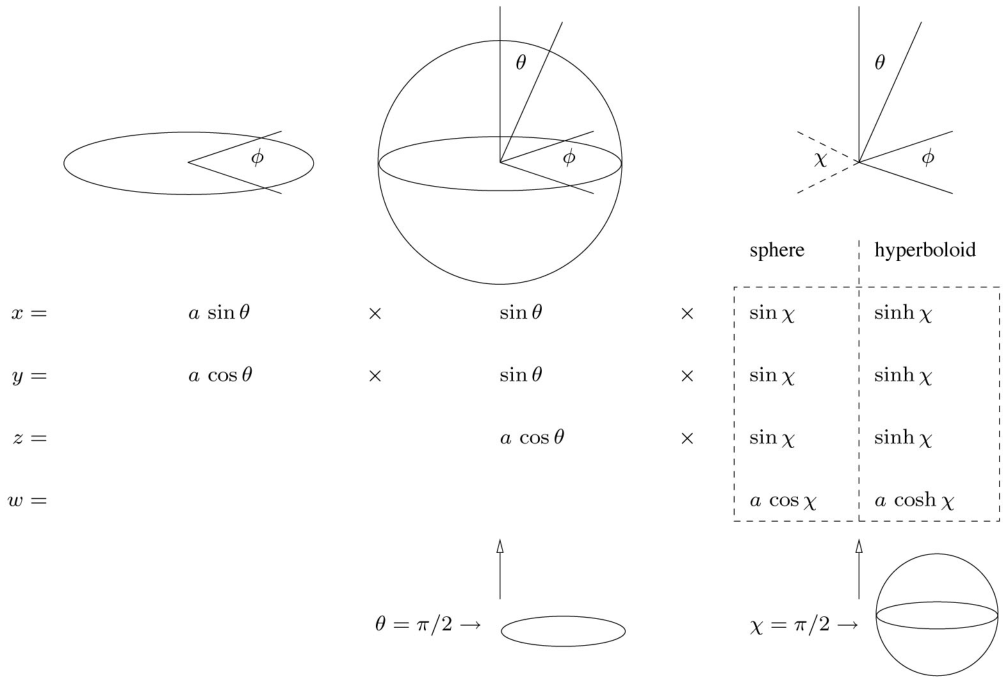


Figure 12: A cheat sheet for spherical and hyperbolical coordinates in four dimensions, see [4].

Finally, we are able to write down the Friedman–Robertson–Walker (FRW) line element

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - k\, r^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right], \tag{2.18}$$

describing a Lorentzian spacetime geometry, $(-,+,+,+)$ with a maximally symmetric three-dimensional subspace with positive eigenvalues and arbitrary curvature. For $k = +1, 0, -1$ it represents spherical, flat, or hyperbolical subspace.

2.2 Equation of state: 1st law of thermodynamics for cosmology (T2)

The expansion of the universe has to affect the energy density content. Consider some volume V with the energy density ρ. The number of particles within are fixed. The first law of thermodynamics relates a small change in the volume to a change in its total energy E,

$$dE = -P\, dV. \tag{2.19}$$

Since $E = \rho V$ and $V = a^3(t)\, V_{\rm coord}$, where $V_{\rm coord} = \text{const}$. We get for the first law of thermodynamics for cosmology

$$\frac{d}{dt}\left[\rho(t)\, a(t)^3\right] = -P(t) \frac{d}{dt}\left[a(t)^3\right]. \tag{2.20}$$

In Figure 13 we have applied this equation to matter and radiation densities. In addition, we briefly discuss the possibility of an constant vacuum density.

	properties	**energy density**
matter	pressureless gass (cosmological dust) $P = 0$	$\frac{d}{dt}\left(\rho_m\, a^3\right) = 0$ $\rightarrow \rho_m(t) = \rho_m(t_0)\left[\frac{a(t_0)}{a(t)}\right]^3$
radiation	blackbody radiation temperature T $P_r = \frac{1}{3}\rho_r$ $\rho_r = g\,\frac{\pi^2}{30}\,\frac{(k_B\,T)^4}{(\hbar\,c)^3}$	$\rho_r(t) = \rho_r(t_0)\left[\frac{a(t_0)}{a(t)}\right]^4$
vaccum	absolute value of vacuum energy density is unknown; we assume that the vacuum energy (i) constant in space and time (ii) positive as indicated by present observations, $P_v = -\rho_v$	$\rho_v = \frac{\Lambda}{8\pi}$ Λ is the csomolgical constant

Figure 13: Energy densities for matter, radiation and vaccum.

Exercise vacuum energy density and CMB temperature. *Use the first law of thermodynamics, see equation (2.20), and the properties for blackbody radiation, that are*

$$P_r = \frac{1}{3} \quad \textit{and} \quad \rho_r = g\frac{\pi^2}{30}\,\frac{(k_B\,T)^4}{(\hbar\,c)^3}\,, \tag{2.21}$$

and derive how the energy density and the CMB temperature scale as a function of the scale factor $a(t)$.

2.3 Dynamics: Friedman equations (T3)

We are left to determine the dependence of the scale factor $a(t)$ on the content of the universe. For that we need to look at the dynamical equations for the FRW spacetime, g_{ab}, such that $ds^2 = g_{ab}\,dx^a dx^b$ corresponds to the line element given in equation (2.18). In general relativity Newton's theory of gravity is replaced by the Einstein equation,

$$G_{ab} + \Lambda\,g_{ab} = 8\pi\,T_{ab}. \tag{2.22}$$

Here G_{ab}, g_{ab} and T_{ab} are rank two covariant tensors and Λ is the cosmological constant.

In general the Einstein equations are rather complicated. Fortunately, for the particular spacetime we are interested in, where g_{ab} is only depending on $a(t)$, the equations simplify tremendously. We can read off g_{ab} from equation (2.18),

$$g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a(t)^2}{1-k\,r^2} & 0 & 0 \\ 0 & 0 & a(t)^2\,r^2 & 0 \\ 0 & 0 & 0 & a(t)^2\,r^2\,\sin^2\theta \end{pmatrix}. \tag{2.23}$$

Notice that G_{ab} on the left-hand side represents the Einstein tensor. The components of the Einstein tensor are functions entirely of of the geometry, thus the components of g_{ab}. In the most general case the components of the Einstein tensor are functions of g_{ab}, and first and second derivatives of g_{ab}, $G_{ab} = G_{ab}(g_{ab}, g_{ab,c}, g_{ab,c,d})$. The right hand side of the Einstein equations is called the stress-energy tensor T_{ab}. The stress-energy tensor contains the sources of the gravitational field, those are the energy densities and pressures of the content of the universe, thus $T_{ab} = T_{ab}(\rho, P)$.

Altogether the Einstein equations relate the following parameter amongst each other:

$$\{ \quad a(t) \quad , \quad \Lambda \quad , \quad \rho \quad , \quad P \quad \}. \tag{2.24}$$

Our task is to determine the scale factor for the matter content of the universe,

$$a(t) = a(\rho(t), P(t), \Lambda). \tag{2.25}$$

Even though we are not familiar with the necessary skills it is straightforward to use Maple and calculate G_{ab} for the metric tensor given in equation (2.26), see appendix A. In orthonormal coordinates we get for the Einstein tensor $G_{\hat{a}\hat{b}}$,

$$G_{\hat{a}\hat{b}} = \begin{pmatrix} 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3\,k}{a^2} & 0 & 0 & 0 \\ 0 & -2\left(\frac{\ddot{a}}{a}\right) - \frac{k}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 & 0 & 0 \\ 0 & 0 & -2\left(\frac{\ddot{a}}{a}\right) - \frac{k}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 & 0 \\ 0 & 0 & 0 & -2\left(\frac{\ddot{a}}{a}\right) - \frac{k}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 \end{pmatrix}. \tag{2.26}$$

In orthonormal coordinates the metric tensor simplifies to

$$g_{\hat{a}\hat{b}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.27}$$

and the stress-energy tensor also simplifies and is given by,

$$T_{\hat{a}\hat{b}} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{2.28}$$

In the orthonormal frame the Einstein equations (2.22) result in a set of coupled differential equaitons,

$$3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3\,k}{a^2} - \Lambda = 8\pi\rho\,, \tag{2.29}$$

$$-2\left(\frac{\ddot{a}}{a}\right) - \frac{k}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 + \Lambda = 8\pi\,P\,. \tag{2.30}$$

A similar, but more detailed, derivation for the Friedman equations is given in Robert Wald's book on "General Relativity" [5]. Notice that it is possible to derive a restricted, $k = 0$, version of the first Friedman equation from Newtonian gravity, for example see James Hartle's book [1].

Exercise Friedman equations. *Show that for*

$$H = \frac{\dot{a}}{a} \tag{2.31}$$

the Friedman equations, see equations (2.29) and (2.30), can be rearranged in the following way

$$H^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{2.32}$$

$$\dot{H} + H^2 = -4\pi\,P - \frac{k}{2\,a^2} - \frac{H^2}{2} + \frac{\Lambda}{2}. \tag{2.33}$$

Notice, that $\rho = \rho(t)$ (total energy density of the universe), $P = P(t)$ (total pressure density of the universe) and thus $H = H(t)$ (Hubble parameter) are changing in time, while the curvature k and the cosmological constant Λ are time-independent. As promised the Friedman equations are dynamical equations for the scale factor in terms of its content.

A. Maple worksheet to calculate Einstein tensor

```
> ##############################################################################
> # Woorksheet to calculate the Einstein tensor, G_ab:
> ##############################################################################
> # Maple has its + T own Tensor toolbox, called 'tensor', wich is called as follows:
> with(tensor);
[Christoffel1, Christoffel2, Einstein, Jacobian, Killing_eqns, Levi_Civita, Lie_diff, Ricci, Ricciscalar, Riemann, RiemannF,     (1)
    Weyl, act, antisymmetrize, change_basis, commutator, compare, conj, connexF, contract, convertNP, cov_diff,
    create, d1metric, d2metric, directional_diff, displayGR, display_allGR, dual, entermetric, exterior_diff,
    exterior_prod, frame, geodesic_eqns, get_char, get_compts, get_rank, init, invars, invert, lin_com, lower, npcurve,
    npspin, partial_diff, permute_indices, petrov, prod, raise, symmetrize, tensorsGR, transform]
> # Let us define our coordiantes as follows:
> coords := [t, r, θ, ϕ];
                                   coords := [t, r, θ, ϕ]                                          (2)
> # To generate the metric tensor in Maple we first have to define an 4x4 array:
> gFRW := array(symmetric, sparse, 1..4, 1..4);
                         gFRW := array(symmetric, sparse, 1..4, 1..4, [])                          (3)
> # So far the array only contains zeros, you may check this by typing "gFRW[i,j]" to the ijth element of the array:
> gFRW[1, 1];
                                           0                                                       (4)
> type(gFRW,'array');
                                          true                                                     (5)
> # Now we put in by hand the non - zero components for the FRW spactime:
> gFRW[1, 1] :=- c^2;
                                    gFRW[1, 1] := -c^2                                             (6)
> gFRW[2, 2] := a(t)^2/(1 - k·r^2);
                                gFRW[2, 2] := a(t)^2/(1 - k r^2)                                   (7)
> gFRW[3, 3] := a(t)^2·r^2;
                                  gFRW[3, 3] := a(t)^2 r^2                                         (8)
> gFRW[4, 4] := a(t)^2·r^2·sin(θ)^2;
                             gFRW[4, 4] := a(t)^2 r^2 sin(θ)^2                                     (9)
> # Finally, we have to tell Maple to use our array gFRW and transform it to a covariant tensor:
> metric := create([-1,-1], eval(gFRW));
```

$$metric := table\left(\left[index\_char = [-1, -1],\ compts = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{a(t)^2}{1-k r^2} & 0 & 0 \\ 0 & 0 & a(t)^2 r^2 & 0 \\ 0 & 0 & 0 & a(t)^2 r^2 \sin(\theta)^2 \end{bmatrix}\right]\right) \quad (10)$$

```
> # Notice, that "create([+1,+1], eval(gFRW))" would create a covariant tensor and thus Maple stores the extra
      information about the idices up or down (+ array) in a table:
> type(metric,'table');
                                          true                                                     (11)
> ##############################################################################
> # This was a lot of work to define the coordinate system and the metric tensor, but now we have everything we
      need to let Maple calculate G_ab.
> # The command to calculate all nessary component is called in the following:
> tensorsGR(coords, metric, contra_metric,'det_met', Ch1, Ch2, Rm, Rc, Rs, G, C);
```

> # Not that we have listed seperately all the "things" we want Maple to calulate. A detailled list of the objects can be found in the help system.

> ##

> # Now let's look at specific results:

> # The Weyl C_{abcd} tensor – Notice that for any FRW spacetime the Weyl tensor must be zero, because it is possible to show that any FRW metric is conformally flat! (Have this in mind for later and for now take it as a consitency check that the line – ement you have put in is correct.)

```
> displayGR(Weyl, simplify(C));
```

The Weyl Tensor

non-zero components :

None

character : [-1, -1, -1, -1] (12)

> # The Rieman tensor R_{abcd}: *(If you want to see the Rieman tensor as well remove "#" in front of the next command line)*

```
> # displayGR(Riemann, simplify(Rm));
```

> # The Ricci tensor R_{ab}: *(If you want to see the Ricci tensor as well remove "#" in front of the next command line)*

```
> # displayGR(Ricci, simplify(Rc));
```

> # The Ricci scalar R:

```
> displayGR(Ricciscalar, simplify(Rs));
```

The Ricci Scalar

$$R = -\frac{6\left(a(t)\left(\frac{d^2}{dt^2}a(t)\right) + k c^2 + \left(\frac{d}{dt}a(t)\right)^2\right)}{a(t)^2 c^2} \tag{13}$$

> # *At an instant of time, when $a(t) = Radius = const$, we get the spacial curvature the Ricci scalar:*

```
> – eval(subs(a(t) = Radius, Rs[compts]));
```

$$\frac{6\,k}{Radius^2} \tag{14}$$

> # *Please notice, that we have multiplied the result with minus one. This is because Maple has chossen a convention cotrarary to the standard notation.*

> # *Standard notation: The curvature for a sphere $(k = 1)$ is positive, for flat space $(k = 0)$ to be zero, and for the hyperploid $(k = -1)$ to be negative.*

> # The Einstein tensor:

```
> displayGR(Einstein, simplify(G));
```

The Einstein Tensor

non-zero components :

$$G11 = -\frac{3\left(\left(\frac{d}{dt}a(t)\right)^2 + k c^2\right)}{a(t)^2}$$

$$G22 = -\frac{2\,a(t)\left(\frac{d^2}{dt^2}a(t)\right) + k c^2 + \left(\frac{d}{dt}a(t)\right)^2}{c^2\left(-1 + k r^2\right)}$$

$$G33 = \frac{r^2\left(2\,a(t)\left(\frac{d^2}{dt^2}a(t)\right) + k c^2 + \left(\frac{d}{dt}a(t)\right)^2\right)}{c^2}$$

$$G44 = -\frac{r^2\left(-2\,a(t)\left(\frac{d^2}{dt^2}a(t)\right) + 2\,a(t)\left(\frac{d^2}{dt^2}a(t)\right)\cos(\theta)^2 - k c^2 + c^2\cos(\theta)^2 k - \left(\frac{d}{dt}a(t)\right)^2 + \cos(\theta)^2\left(\frac{d}{dt}a(t)\right)^2\right)}{c^2}$$

$$\text{character} : [-1, -1] \tag{15}$$

\> # In orthnomarl coordinates we get:

\> frame(metric, inv_tetrad, η , coords);

\> eval(η):

\> tetrad := invert(inv_tetrad,'det_tetrad'):

\>

\>

\> GG := change_basis(G, tetrad, inv_tetrad):

\> # we get:

\> Gtt := simplify(GG[compts][1, 1],'assume=real')

$$Gtt := -\frac{3\left(\left(\frac{d}{dt}a(t)\right)^2 + k c^2\right)}{c^2 a(t)^2} \tag{16}$$

\> Gxx := simplify(GG[compts][2, 2],'assume=real')

$$Gxx := -\frac{\left(2 a(t)\left(\frac{d^2}{dt^2}a(t)\right) + k c^2 + \left(\frac{d}{dt}a(t)\right)^2\right)\left|\frac{-1 + k r^2}{a(t)^2}\right|}{c^2(-1 + k r^2)} \tag{17}$$

\> Gyy := simplify(GG[compts][3, 3],'assume=real')

$$Gyy := \frac{2 a(t)\left(\frac{d^2}{dt^2}a(t)\right) + k c^2 + \left(\frac{d}{dt}a(t)\right)^2}{|a(t)|^2 c^2} \tag{18}$$

\> Gzz := simplify(GG[compts][4, 4],'assume=real');

$$Gzz := \frac{2 a(t)\left(\frac{d^2}{dt^2}a(t)\right) + k c^2 + \left(\frac{d}{dt}a(t)\right)^2}{|a(t)|^2 c^2} \tag{19}$$

\> # Notice that for $k=1$, we have to assume $1 - kr^2 > 0$, also that $a(t)^2 > 0$, such that $Gxx = Gyy = Gzz$

\> ###

\> # The final results are given by:

\> # In the new coordinates we get $G_{nn} := -G_{xx} = -G_{yy} = -G_{zz}$:

\> Gnn := − Gzz;

$$Gnn := -\frac{2 a(t)\left(\frac{d^2}{dt^2}a(t)\right) + k c^2 + \left(\frac{d}{dt}a(t)\right)^2}{|a(t)|^2 c^2} \tag{20}$$

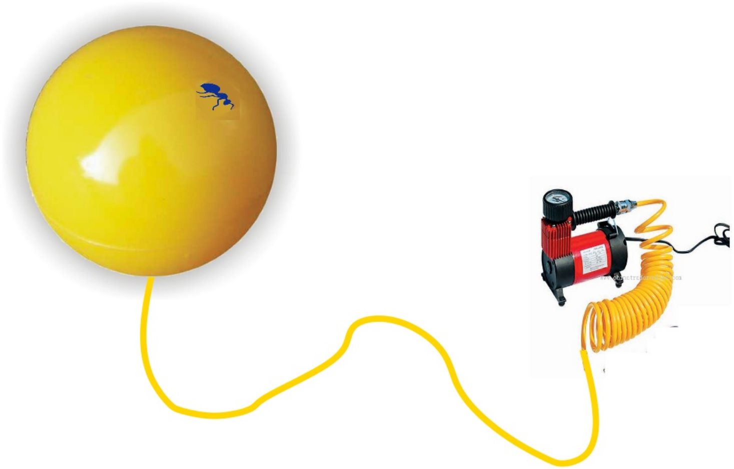
\> # Also the Gtt component has to be multiplied with −1 :

\> Gtt := −G[compts][1, 1];

$$Gtt := \frac{3\left(\left(\frac{d}{dt}a(t)\right)^2 + k c^2\right)}{a(t)^2} \tag{21}$$

\> ###

B. A little toy model



Exercise: Blue ant living on a yellow balloon. *Let us assume a blue ant is bound to live on a yellow balloon. The ant is two-dimensional, blind, very smart and the only way it can communicate is by sending little surface waves along the surface of the balloon. The surface waves propagate with a constant velocity* c_s*. The yellow balloon is hooked up to an electronic air-pump and consists of a magic material that can stretch infinitely. (The surface waves always remain constant independently of the expansion of the balloon.)*

(1) Derive the FRW line element for the ant on the balloon assuming that the ant cannot run faster than c_s*.*

(2) Now imagine you are in control of the air-pump. The air-pump has little adjustable electronic device that allows you to control the air-stream into the balloon. Relate the air-stream (not necessarily constant) with a change in the scale factor for our little toy universe and calculate the air-stream necessary to have $a(t) = \exp(H_0\, t)$*, where* H_0 *is the expansion rate for our universe.*

(3) What are the key differences between our little toy model and the real universe?

References

[1] J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity.* Addison Wesley, December, 2002.

[2] S. Dodelson, *Modern cosmology.* Academic Press, 2003. 440 pp.

[3] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, vol. 2008. Wiley, 1972.

[4] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* W. H. Freeman, September, 1973.

[5] R. M. Wald, *General Relativity.* Chicago university press, Chicago, 1984.