Physics 200-04

Two Two level systems

While the two level system is interesting, it eventually gets boring. Although at least two Nobel prizes have been awarded for understanding the two level system, let us now look at a slightly more complicated system, two two level system.

We assime that we have two separate two level systems. Each has physical attributes which can take only two values, and has operators-matrices– which represent those two values. Let us choose those two values for some attribute to be ± 1 for each of the systems, and let the matrices which represent that attribute be σ_3 for each system. We need a notation which will differentiate between thse two σ_3 operators. I will choose loser case and capital. Ie, σ_3 will be the attribute for one of the systems, and Σ_3 the attribute for the other. The totality of Pauli spin matrices are then designated by $\sigma_1, \quad \sigma_2, \quad \sigma_3$ for the one and $\Sigma_1, \quad \Sigma_2, \quad \Sigma_3$ for the other. There will now be four possible states of the system– one with eigenvalue $+1$ for σ_3 and $+1$ for Σ_3, $+1$ for σ_3 and -1 for Σ_3, -1 for σ_3 and $+1$ for Σ_3 and finally -1 for σ_3 and -1 for Σ_3.

We will designate the eigenstates in the four dimensional vector space by $|s, S\rangle$ where s and S both take values of ± 1. We will also designate this same vector by $|s\rangle|S\rangle$.Note that this is NOT ordinary matrix multiplication, since you cannot multiply a column vector by a column vector.This notation is to emphasise that the two column vectors refer to two separate systems, the lower case and upper case system. In each case, the lower case operators multiply only the first ket, and the upper case operators multiply only the second ket. Thus

$$\sigma_1|1,1\rangle \equiv \sigma_1(|1\rangle|1\rangle) = (\sigma_1|1\rangle)|1\rangle = |-1\rangle|1\rangle = |-1,1\rangle \tag{1}$$

is the definition of how the σ_1 operator alters this particular state of the system.

We can also do the same for our other operators.

This system can be represented by matrices as well, four dimensional (3 rows) for the ket vectors, and 4x4 for the matrices themselves. These are called the direct product matrices. However, it is rarely useful to do so. The matrices simply do not give any clue as to what the individual elements of the matrix refer to– they do not preserve the distinction between the two systems which we want to preserve. Thus, instead of writing out the various

terms as matrices, we will work with the more abstract expressions using the bra-ket notation as above. You can keep in your mind that these really refer to matrices in some abstract sense, but worrying about what the matrix looks like is usually not very rewarding.

The notation for the operators will also be a bit confusing. Let us say that we want to know how the attribute of first doing σ_1 on a ket and then Σ_2 would on a ket would be represented. We represent it as though it were a product, $\Sigma_2\sigma_1$ but this is not to be regarded as the product of the two matrices which represent the two attributes. They operate on different vectors. Thus

$$\Sigma_2\sigma_1|1,1\rangle = (\sigma_1|1\rangle)(\Sigma_2|1\rangle) = (|-1\rangle)(i|-1\rangle) = i|-1,-1\rangle \tag{2}$$

recalling that the second position is for the second capital particle, and the first is for the first particle. Note that the constants in the product do multiply the whole of the vector. The inner product is the combination of the two inner products of the two systems. Thus if we have a vector $|\psi,\phi\rangle = |\psi\rangle|\phi\rangle$ then the inner product is

$$\langle\psi,\phi||\psi,\phi\rangle = \langle\psi||\psi\rangle\langle\phi||\phi\rangle \tag{3}$$

Sometimes to emphasise the differences between the various vectors, one puts subscripts on them. Thus one could write the above as

$$\langle\psi,\phi||\psi,\phi\rangle = \langle\psi|_1|\psi\rangle_1\langle\phi|_2|\phi\rangle_2 \tag{4}$$

to emphasise that those vectors are for the first and second particles respectively.

Just as for the simple, single system, one allows linear combinations. Thus, we have for example

$$|1,1\rangle + |1,-1\rangle = |1\rangle_1|1\rangle_2 + |1\rangle_1|-1\rangle_2 \tag{5}$$

We can also write this as

$$|1,1\rangle + |1,-1\rangle = |1\rangle_1(|1\rangle_2 + |-1\rangle_2) \tag{6}$$

as if this multiplication of vectors corresponding to two different particles were ordinary multiplication. (It is under a suitable definition of "direct product" multiplication).

In part this definition preserves the independence of the two particles. Something done to one of the particles does not affect the other particle.

In order to preserve the inner product (ie, orthogonal vectors are taken to orthogonal vectors) which also preserves the Hermitean nature of the operators and attributes (the values of the operator remain the same under a transformation), the transformation on this "product" vector space must again be unitary transformations. $UU^\dagger = I$.

Clearly any unitary transformation on any one of the particles is still a unitary transformation on the whole. Thus $U|\psi,\phi\rangle \equiv (U_1|\psi\rangle)(U_2|\phi\rangle)$ will be a unitary transformation if U_1 and U_2 are on their respective particles. There are however unitary transformations which mix the two particles. For example, consider the transformation which exchanges the two particles.

$$U|\psi,\phi\rangle = |\phi,\psi\rangle \tag{7}$$

This clearly takes orthogonal vectors into orthogonal vectors. However, it just as clearly is not a unitary transformation which can be written as a product transformation of two individual transformations on each of the particles separately.

Bell's Theorem

Consider the state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|+1,-1\rangle - |-1,+1\rangle)$ and consider the the two operators $A = \vec{A}\cdot\vec{\sigma}$ and $B = \vec{B}\cdot\vec{\Sigma}$. A is an matrix for the first particle only and B is for the second. I do not use subscripts as this could confuse between the components of the vector $\vec{A}$ of numbers in the expansion of A in terms of the sigma matrices, and the first or second particle. Remember in the following that A is a matrix representing an attribute of the first particle only, and B is an attribute of the second particle. Thus AB is not the ordinary product of the two matrices– rather it is the direct product $A \otimes B$

Now consider the expectation value of the product of these

$$\begin{aligned}
\langle\Psi|AB|\Psi\rangle &= \frac{1}{2}\left((\langle 1|_1\langle -1|_2 - \langle -1|_1\langle 1|_2)AB(|1\rangle_1|-1\rangle_2 - |-1\rangle_1|1\rangle_2)\right) \\
&= \frac{1}{2}\langle 1|_1\langle -1|_2 AB|1\rangle_1|-1\rangle_2 - \frac{1}{2}\langle 1|_1\langle -1|_2 AB|-1\rangle_1|1\rangle_2 \\
&\quad -\frac{1}{2}\langle -1|_1\langle 1|_2 AB|1\rangle_1|-1\rangle_2 + \frac{1}{2}\langle -1|_1\langle 1|_2 AB|-1\rangle_1|1\rangle_2 \\
&= \frac{1}{2}\langle 1|_1 A|1\rangle_1\langle -1|_2 B|-1\rangle_2 - \frac{1}{2}\langle 1|_1 A|-1\rangle_1\langle -1|_2 B|1\rangle_2
\end{aligned}$$

$$-\frac{1}{2}\langle -1|_1 A|1\rangle_1 \langle 1|_2 B|-1\rangle_2 + \frac{1}{2}\langle -1|_1 A|-1\rangle_1 \langle 1|_2 B|1\rangle_2 \tag{8}$$

Now,

$$\langle 1|_1 A|1\rangle_1 = -\langle -1|_1 A|-1\rangle_1 = A_3 \tag{9}$$

and

$$\begin{aligned} \langle 1|_1 A|-1\rangle_1 &= \langle -1|_1 A|1\rangle_1^* = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} A_3 & A_1 - iA_2 \\ A_1 + iA_2 & -A_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= A_1 - iA_2 \end{aligned} \tag{10}$$

The same thing will be true for $B = \vec{B} \cdot \vec{\Sigma}$. Thus, we have

$$\begin{aligned} \langle \Psi|AB|\Psi\rangle &= \frac{1}{2}\left(A_3(-B_3) - (A_1 - iA_2)(B_1 + iB_2) - ((A_1 + iA_2)(B_1 - iB_2) + (-A_3)(B_3)\right) \\ &= -(A_1 B_1 + A_2 B_2 + A_3 B_3) = -\vec{A} \cdot \vec{B} \end{aligned} \tag{11}$$

no matter which direction $\vec{A}$ and $\vec{B}$ point in.

Choose $\vec{A}$ and $\vec{B}$ to be unit vectors– ie $\vec{A} \cdot \vec{A} = 1$, and similarly for $\vec{B}$. Then the eigenvalues of A and B are ± 1.

Ie, in any determination of A and B only the values of 1 or -1 will be obtained.

The correlation function betweent the values of A and B is thus just the minus the cosine of the angle between the vectors $\vec{A}$ and $\vec{B}$.

This is the quantum mechanical result.

Classical

Bell (well, Clauser, Horne, Shimony and Hall who simplified Bell) argues as follows. Assume in classical physics that we have two attributes A and C on particle 1 both of which can only have two values, ± 1. Similarly consider that there are two attributes on particle 2, B and D, both of which again can have only values ± 1. Assume that the values of these various variables are determined by some other classical hidden variable λ. λ need not be some single cause, it could be as complicated as you like. Assume forthermore, that we allow a statistical theory, so that λ has some probability distribution $P(\lambda)$ which can be arbitrary, except that it is always positive and that it integrates to unity (both requirements of what one would call a probability

distribution) Ie, the probability of λ having some given value is a positive number (since we have no idea what a negative probability would mean) and that lambda must have some value.

Now, we assume that the values of A,B,C and D are determined by λ in some way. Ie, for each value of λ, $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ all have some value, in each case ± 1. I will not assume that I can measure them all. It may be impossible for some reason that I can actually physically measure A abd C at the same time, and similarly for B and D.

Now consider $(A+C)B+(A-C)D$ for any value of λ. Let us say that the values of A and C are the same. Then the term multiplying D will be 0, and this expression will have a value or either +2 or -2. On the other hand, for that given value of λ $A(\lambda)$ and $B(\lambda)$ could be different. In that case $(A+B)$ will be zero, $A-B$ will be ± 2 as will $D(A-C)$. Ie, for any value of λ that expression will either have value of $+2$ or -2. We now average over all possible values of λ. That average must lie between -2 and $+2$.

By assumption, we cannot, for some reason or another, ever measure A and C at the same time, nor B and D. However, separate the two particles by a long long distance, and allow each observer to randomly choose which of A, C and B, D to measure. Once each has made many many measurements of new, identically prepared systems, we can calculate the correlation functions $< AB >$, $< AD >$, $< CB >$, and $< CD >$. Ie for each pair in which A was measured on the first particle and B on the second, one multiplies those two values together and averages them. Similarly for each of the other correlations. We now calculate

$$Corr =< AB > + < CB > + < AD > - < CD > \tag{12}$$

which should be a good estimate of the expectation of $(A+C)B+(A-C)D$, even though we have not measured the this product on any single system.

This correlation function should therefor lie between -2 and 2. This is Bell's theorem. Note that it is an incredibly powerful theorem. It asumes nothing about the dynamics of the particles–their laws of motion. It simply assumes that the values are determined by some variables λ. These could be the initial values in the past or whatever.

Now comes the *pièce de résistance*. In quantum mechanics choose the vector A so that $A_1 = 1,\ A_2 = A_3 = 0,\ B_1 = \frac{1}{\sqrt{2}},\ B_2 = 0,\ B_3 = \frac{1}{\sqrt{2}},$ $C_1 = C_2 = 0,\ C_3 = 1$ and $D_1 = \frac{1}{\sqrt{2}},\ D_2 = 0,\ D_3 = -\frac{1}{\sqrt{2}}$ Then in the above

state,

$$\begin{aligned} < AB >= -\vec{A} \cdot \vec{B} = -\frac{1}{\sqrt{2}} \\ < CB >= -\vec{C} \cdot \vec{B} = -\frac{1}{\sqrt{2}} \\ < AD >= -\vec{A} \cdot \vec{D} = -\frac{1}{\sqrt{2}} \\ < CD >= -\vec{C} \cdot \vec{D} = \frac{1}{\sqrt{2}} \end{aligned} \tag{13}$$

Thus quantum mechanically, we have

$$< AB > + < AD > + < CB > - < CD >= -\frac{4}{\sqrt{2}} = -2\sqrt{2} < -2 \tag{14}$$

Ie, the quantum anti-correlation is stronger that it is possible for any classical correlation to ever be. Thus it is impossible to describe the quantum system in terms of any hidden variables theory.

There are two ways out of this conclusion. The first is that the value of λ not only determines the values of the variables A, B, C, D but also the choice of the experimentalist– Ie, the experimentalist will always choose which of the variables to measure, based on the value of λ such that the correlations come out right. This of course implies a much much worse conspiracy in the world, a conspiracy which moreover makes physics almost impossible– systems, including the system which is the experimentalists, must all be highly interrelated. You cannot ever make the approximation of separate systems.

The second way out is that somehow the measurement of say A on the one particle influences the outcome of the measurement of say B on the other. Somehow the second particle knows what measurement and what outcome was made on the second particle in just such a way as to increase the anticorrelation between the values of A and B. This would have to be true even if the two particles were separated by arbitrary far distance and if the particles decisions as to which measurements to be made were made at times with are spacelike separated. Ie, the classical description could be resucued if one threw out all notions of causality.

It is crucial to notice that this argument says nothing about quantum mechanics and causality. It simply says that IF you want to make quantum

mechanics into a deterministic theory dependent on some, at present,1 hidden variables, then that theory must also non-causal. Things here must be able to influence things there over spacelike separated distances. It does not say that quantum mechanics, which is not a hidden variables theory, is non-local. Even people who should know better sometimes talk as if it is.