Physics 200-05
Assignment 7

1. Show explicitly that the eigenvalues for the matrix $A = a_0 I + \vec{a}\cdot\vec{\sigma}$ are $a_0 \pm \sqrt{\vec{a}\cdot\vec{a}}$.

$$\begin{pmatrix} a_0+a_3 & a_1-ia_2 \\ a_1+ia_2 & a_0-a_3 \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \lambda\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$
$$(a_0+a_3)\alpha + (a_1-ia_2)\beta = \lambda\alpha$$
$$(a_1+ia_2)\alpha + (a_0-a_3)\beta = \lambda\beta \quad (1)$$

Solving the second for α and substituting into the first we get

$$\frac{(a_0+a_3)(-(a_0-a_3)+\lambda)}{(a_1+ia_2)} + (a_1-ia_2) = \lambda\frac{(-(a_0-a_3)+\lambda)}{(a_1+ia_2)} \quad (2)$$

Solving for λ we get

$$\lambda = a_0 \pm \sqrt{a_1^2+a_2^2+a_3^2} \quad (3)$$

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If

$$a_1 = a\sin(\theta)\cos(\phi)$$
$$a_2 = a\sin(\theta)\sin(\phi)$$
$$a_3 = a\cos(\theta) \quad (4)$$

then show that the eigenvector for the $a_0 + a$ eigenvalue is

$$|a_0+a\rangle = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix} \quad (5)$$

and for the other eigenvalue the eigenvector is

$$|a_0-a\rangle = \begin{pmatrix} -e^{-i\phi}\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{pmatrix} \quad (6)$$

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\begin{aligned}
& \begin{pmatrix} a_0 + a\cos(\theta) & a(sin(\theta)(\cos(\phi) - i\sin(\phi)) \\ a(sin(\theta)(\cos(\phi) + i\sin(\phi) & a_0 - a\cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix} \\
= & \begin{pmatrix} (a_0 + a\cos(\theta))\cos(\frac{\theta}{2}) + a(sin(\theta)(\cos(\phi) - i\sin(\phi))e^{i\phi}\sin(\frac{\theta}{2}) \\ a(sin(\theta)(\cos(\phi) + i\sin(\phi)\cos(\frac{\theta}{2}) + (a_0 - a\cos(\theta))e^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix} \\
= & \begin{pmatrix} a_0\cos(\frac{\theta}{2} + a(cos(\theta)cos(\frac{\theta}{2}) + sin(\theta)\sin(\frac{\theta}{2})) \\ e^{i\phi}(a_0 sin(\frac{\theta}{2} + a(sin(\theta)\cos(\frac{\theta}{2}) - \cos(\theta)sin(\frac{\theta}{2})) \end{pmatrix} \\
= & (a_0 + a)\begin{pmatrix} \cos(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix} \qquad (7)
\end{aligned}
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Ie, this is the eigenvector for the eigenvalue $a_0 + a$.

(I used the fact that $cos(\phi) \pm isin(\theta) = e^{\pm i\phi}$ and the difference of angles formulas for the trigonometric functions.)

One can do exactly the same multiplication to show that the otherone is the eigenvector for $a_0 - a$ or one can remember that the eigenvectors for the two eigenvalues are orthogonal to each other.

$$\langle a_0 + a||a_0 - a\rangle = 0$$

so we just have to check that this is true to see that this must be the eigenvector for $a_0 - 1$.

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\begin{aligned}
\langle a_0 + a||a_0 - a\rangle &= \begin{pmatrix} \cos(\frac{\theta}{2}) & e^{-i\phi}\sin(\frac{\theta}{2}) \end{pmatrix} \begin{pmatrix} -e^{-i\phi}\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{pmatrix} \\
&= e^{-i\phi}(-cos(\theta/2)sin(\theta/2) + sin(\theta/2)cos(\theta/2) = 0 \quad (8)
\end{aligned}
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2. Consider the state vector

$$|\psi\rangle = \frac{1}{\sqrt{(2)}}\begin{pmatrix} 1 \\ \frac{1+i}{\sqrt{2}} \end{pmatrix} \qquad (9)$$

a)What is the unit vector $|\phi\rangle$ orthogonal to this vector? Ie, $\langle\phi||\psi\rangle = 0$?

$$\langle\psi|\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}}(\alpha + \frac{(1-i)}{\sqrt{2}}\beta) = 0 \qquad (10)$$

The easiest way to do this is simply to interchange the two components and stick in a minus sign.

$$\alpha = \frac{1}{\sqrt{2}}\frac{(1+i)}{\sqrt{2}}$$

$$\beta = -\frac{1}{\sqrt{2}}$$

Ie, if one has a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ then $\begin{pmatrix} b^* \\ -a^* \end{pmatrix}$ is clearly orthogonal, and furthermore has the same magnitude.

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b) Show that the matrix $A = |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|$ has eigenvalues ± 1 and eigenvectors $|\psi\rangle$ and $|\phi\rangle$. (Remember that $|\mu\rangle\langle\nu|$ is the product of a column vector times a row matrix, which is a 2x2 matrix if the $|\mu\rangle$ and $|\nu\rangle$ are 1x2 vectors.)

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} \quad (11)$$

$$A|\psi\rangle = (|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)|\psi\rangle = |\psi\rangle\langle\psi||\psi\rangle - |\phi\rangle\langle\phi||\psi\rangle = |\psi\rangle - 0 = |\psi\rangle \quad (12)$$

Similarly

$$A|\phi\rangle = (|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)|\phi\rangle = |\psi\rangle\langle\psi||\phi\rangle - |\phi\rangle\langle\phi||\phi\rangle = 0 - |\phi\rangle = -|\phi\rangle \quad (13)$$

since $\langle\psi||\phi\rangle = \langle\phi||\psi\rangle = 0$ and $\langle\psi||\psi\rangle = \langle\phi||\phi\rangle = 1$

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Finally show that $|\psi\rangle\langle\psi|$ is a projection operator (has a single eigenvalue of value 1 and the other eigenvalue has value 0) with $|\psi\rangle$ as the eigenvector with 1 as the eigenvalue.

This follows almost immediatly from the above.

$$|\psi\rangle\langle\psi||\psi\rangle = |\psi\rangle \quad (14)$$

so $|\psi\rangle$ is an eigenvector with eigenvalue 1. Similarly

$$|\psi\rangle\langle\psi||\phi\rangle = 0 \quad (15)$$

so $|\phi\rangle$ is an eigenvector with eigenvalue 0.
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3. Given the matrix

$$A = \begin{pmatrix} 3 & 2+2i \\ 2-2i & -1 \end{pmatrix} \tag{16}$$

what are the values of $a_0,\ a_1,\ a_2,\ a_3$ and what are the eigenvalues of this matrix?

$$a_0 = 1,\ a_3 = 2,\ a_1 = 2,\ a_2 = -2$$

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What is the projection matrix onto the larger eigenvalue? If the state $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ what is the probability that the largest eigenvalue of A is obtained in a measurement.

The Projection matrix is $\frac{1}{2}(I + \frac{\vec{a}}{\sqrt{\vec{a}\cdot\vec{a}}} \cdot \vec{\sigma}$. $\vec{a}\cdot\vec{a} = 2^2 + 2^2 + 2^2 = 12$ so the projection matrix is

$$P_+ = \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(\sigma_1 - \sigma_2 + \sigma_3)\right) = \frac{1}{2}\begin{pmatrix} 1+\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} + \frac{i}{\sqrt{3}} \\ 1 - \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} - \frac{i}{\sqrt{3}} \end{pmatrix} \tag{17}$$

The probability of measuring the eigenvalue whose eigenvector is $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is

$$\langle\psi|A|\psi\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1+\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} + \frac{i}{\sqrt{3}} \\ 1 - \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} - \frac{i}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}(1 + \frac{1}{\sqrt{3}}) \tag{18}$$

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4) Show that

$$[A, BC] = [A, B]C + B[A, C] \tag{19}$$

where $A,\ B,\ C$ are matrices and $[A, B] = AB - BA$ is the commutator.

$$[A, BC] = ABC - BCA = ABC - BAC + BAC - BCA = [A, B]C + B[A, C] \tag{20}$$

as required

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Show that if X and P obey

$$[X,P] = i\hbar I \tag{21}$$

and if we define the Energy as

$$H = \frac{1}{2m}P^2 + \frac{k}{2}X^2 \tag{22}$$

where m and k are real numbers. Then

$$[X,H] = i\hbar\frac{1}{m}P \tag{23}$$

and

$$[P,H] = -i\hbar kX \tag{24}$$

$$\begin{aligned}[X,H] &= [X,\frac{1}{2m}P^2 + \frac{k}{2}X^2] = [X,\frac{1}{2m}P^2] + \frac{k}{2}[X,X^2] \\ &= \frac{1}{2m}([X,P]P + P[X,P]) + 0 = \frac{i\hbar}{2m}(IP + PI) = \frac{i\hbar}{m}P \end{aligned} \tag{25}$$

and

$$[P,H] = [P,\frac{1}{2m}P^2 + \frac{k}{2}X^2] = [P,\frac{1}{2m}P^2] + \frac{k}{2}[P,X^2] = 0 + \frac{k}{2}([P,X]X + X[P,X]) = -i\hbar kX \tag{26}$$

since $[P,X] = -[X,P]$

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Show that if we define the non-Hermitean matrix

$$A = (km)^{\frac{1}{4}}X + i\frac{1}{(km)^{\frac{1}{4}}}P$$

, then

$$[A,H] = \hbar\sqrt{\frac{k}{m}}A \tag{27}$$

$$\begin{aligned}[A,H] &= (km)^{\frac{1}{4}}[X,H] + i\frac{1}{(km)^{\frac{1}{4}}}[P,H] = i\hbar\frac{(km)^{\frac{1}{4}}}{m}P - i\hbar k i\frac{1}{(km)^{\frac{1}{4}}}X \\ &= \hbar\sqrt{\frac{k}{m}}(i\frac{1}{(km)^{\frac{1}{4}}}P + (km)^{\frac{1}{4}}X) = \hbar\sqrt{\frac{k}{m}}A \end{aligned} \tag{28}$$

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Finally, show that if $|E\rangle$ is an eigenvector if H with eigenvalue E, then $A|E\rangle$ is an eigenvector of H with eigenvalue $E - \hbar\sqrt{\frac{k}{m}}$. A is called the annihilation operator for the simple harmonic oscillator because it annihilates one unit of energy. (Ie, if the state has energgy E, the new state after operating on it by A has one unit less energy)

Assume that $|E\rangle$ is the eigenvector for H so $H|E\rangle = E|E\rangle$. Then

$$\begin{aligned}H(A|E\rangle) &= H(A|E\rangle) - AH|E\rangle + AH|E\rangle = [H,A]|E\rangle + AE|E\rangle \\ &= -\hbar\sqrt{\frac{k}{m}}A|E\rangle + EA|E\rangle = (E - \hbar\sqrt{\frac{k}{m}})|E\rangle \end{aligned} \tag{29}$$

Ie, $A|E\rangle$ is the eigenvector of H with eigenvalue $E - \hbar\sqrt{\frac{k}{m}}$.

The following is an addition whichis not part of the solution:

Since both $\langle E|P^2|E\rangle = (P|E\rangle)^\dagger P|E\rangle > 0$ and similarly $\langle E|X^2|E\rangle > 0$ we have that $\langle E|H|E\rangle = E\langle E||E\rangle > 0$ the energy must always be greater than 0. Thus eventually $E - n\hbar\sqrt{\frac{k}{m}}$ must go negative for big enough n. Thus unless at some point $A^n|E\rangle$ must be zero. Ie, if E_0 is the smallest eigenvalue for H, then $A|E_0\rangle = 0$. Ie there MUST be a smallest energy and all of the larger energies must be larger.

One can show that $H = \frac{1}{2}\sqrt{\frac{k}{m}}A^\dagger A + AA^\dagger)$ and that $[A, A^\dagger] = -\hbar I$ finally giving $H = \sqrt{\frac{k}{m}}(A^\dagger A + \frac{1}{2}\hbar$ Thus applying H to that minimum eigenvector, we have

$$H|E_0\rangle = \frac{1}{2}\sqrt{\frac{k}{m}}|E_0\rangle$$

Ie the minimum energy is $E_0 = \frac{1}{2}\sqrt{\frac{k}{m}}$, and all of the higher energies must be of the form $(n + \frac{1}{2})\sqrt{\frac{k}{m}}$

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This can be used to show that the eigenvalues for the Harmonic oscillator must have values $(n + \frac{1}{2})\hbar\sqrt{\frac{k}{m}}$ where n is a positive integer. (You do not have to show this, but if you want to do it for yourself, The key is that there must be a minimum eigenvalue since $\langle\psi|H|\psi\rangle$ is greater than 0 and thus the A cannot step the eigenvalues to less than 0.)

(See the text book for further explication.)