Physics 200-06
Assignment 6

1) A particle is found by measurement to have the the value +2 for the physical attribute represented by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. What is the probability that if the physical attribute represented by the $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ matrix is measured, its value is found to be the largest value of this attribute.

We need to find the eigenstate for the +2 eigenvalue for the first matrix. Ie if $|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ then we want

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 2 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{1}$$

or

$$\psi_1 + \psi_2 = 2\psi_1 \tag{2}$$
$$\tag{3}$$

which gives $\psi_1 = \psi_2$. The vector is thus a multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or normalizing $|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The eigenvalue equation for $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ with eigenvalue λ and eigenvector $\begin{pmatrix} a \\ b \end{pmatrix}$ is

$$a + b = \lambda a \tag{4}$$
$$a = \lambda b \tag{5}$$

or $\lambda + 1 = \lambda^2$ which gives $\lambda = \frac{1}{2}(1 \pm \sqrt{1+4})$ The largest eigenvalue is $\frac{1}{2}(1+\sqrt{5})$. Thus we have $a = \frac{1}{2}(1+\sqrt{5})b$ or $\begin{pmatrix} a \\ b \end{pmatrix} = b \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix}$.

The probability is

$$(\begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & 1 \end{pmatrix} |\psi\rangle)^2 / \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix} \tag{6}$$

$$= \frac{(\frac{1}{\sqrt{2}}(\frac{1}{2}(3+\sqrt{5}))^2}{(3+\sqrt{5})/2} \tag{7}$$

$$= (\frac{1}{2} + \frac{1}{\sqrt{5}}) \tag{8}$$

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2) A particle is found by measurement to have the value of +1 for the attribute represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the attribute $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is measured and found to have value +2. What is the probability that if $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is remeasured, its value is found to be -1?

The key is that after the second measurement, the first state if forgotten. Thus the fact that the first eigenvalue of the first matrix was obtained is irrelevant. Thus all we need to do is find the eigenvector corresponding to the +2 eigenvalue of the second matrix. It is clearly $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The third matrix is just σ_1 It has eigenvectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for the +1 eigenvalue and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for the -1 eigenvector. Thus the probability of the second measurement is

$$P = |\begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}|^2 = \frac{1}{2} \tag{9}$$

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3) The probability that it will rain today is .4 and tomorrow it is .6.

i)What is the probability that it will not rain today?

Since it will either rain today or not, the probability of rain plus the probability of no rain is 1. Thus the probability of no rain today is .6.

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ii)What is the probability that it will rain both today and tomorrow?

Lets say that we have 100 cases. In 40 cases it will rain today. Of those in .6 times 40 cases it will also rain tomorrow, or in 24 cases. Thus the probability of rain today and tomorrow is 24/100=.24=.6 .4.

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iii) If it rains today, the amount of rain will be 12mm. What is the average rainfall predicted for today?

Again in 100 days just like today, in 40 it will rain 12 mm on each or a total 480mm altogether. In 60 days it will not rain at all. Thus the average rainfall would be 480/100=4.8mm of rain.
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4) The following is a table of the number of times, in a throwing of a dice 1000 times, that the faces with the various numbers of spots comes up with the various values. What are the probabilities that two of the same dice will come up with their spots totalling 12? Totalling 7? (Assume, contrary to fact, that the probabilities are accurately reflected by their frequencies).

Spots	#times
1	155
2	158
3	175
4	170
5	180
6	162

(10)

There is onlyone way of coming up 12 and that is come up 6 on each throw. The probability is .162x.162=.026

To roll a 7, there are 7 ways, 1 on the first roll, and 6 on the second, 2 on the first and 5 on the second, 3 and 4, 4 and 3, 5 and 2 and 6 and 1.

Thus the chances of a 7 are .155 .162+.158 .180+ .175 .170 +.170 .175+ .180 158 + .162 .155 = .167
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5) i)Show that any 2x2 Hermitean matrix which is not a multiple of the identity matrix has two distinct eigenvalues.

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

or

$$a\alpha + b\beta = \lambda\alpha \tag{11}$$

$$b^*\alpha + c\beta = \lambda\beta \tag{12}$$

or solving the second for α in terms of β and substituting into the first, we have

$$a\frac{\lambda-c}{b^*}+b=\lambda\frac{\lambda-c}{b^*}$$

or

$$\lambda^2-(a+c)\lambda-(bb^*-ac)=0$$

or

$$\lambda=(a+c)/2\pm\sqrt{(a+c)^2+4b^*b-4ac}/2=(a+c)/2\pm\sqrt{(a-c)^2+4b^*b}$$

The only case where there is only one eigenvalue is if the argument of the sqare root is zero. But it is zero only if b=0 and a=c, in which case the matrix is a multiple of the identity.

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ii) Show that the non-Hermitean matrix $\begin{pmatrix}0&1\\0&0\end{pmatrix}$ has only one eigenvector (up to multiplications by a constant) and one eigenvalue.

This is NOT a Hermitean matrix. We have

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}=\lambda\begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

or

$$\beta=\lambda\alpha \tag{13}$$

$$0=\lambda\beta \tag{14}$$

The second equation says that either β is 0 or λ is 0. If lambda is 0, then beta is also zero, and the eigenvector has only α is non-sero. If λ is not zero, then beta is zero, and from the first α is also zero and there is no eigenvector. Thus the only eigenvalue is $\lambda=0$ and the eigenvector is $\begin{pmatrix}1\\0\end{pmatrix}$.

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6)

$$|\phi\rangle=\begin{pmatrix}1\\1+2i\end{pmatrix} \tag{15}$$

$$|\psi\rangle=\begin{pmatrix}3+4i\\5i\end{pmatrix} \tag{16}$$

What are the normalized vectors corresponding to these two vectors? What is $\langle\phi||\psi\rangle$ and $\langle\psi||\phi\rangle$?

$$\langle\phi||\phi\rangle = \begin{pmatrix} 1 & 1-2i \end{pmatrix} \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} = 1+1+4 = 6$$

Thus the normalisation would be to divide $|\phi\rangle$ by $\sqrt{6}$

For ψ we have

$$\langle\psi||\psi\rangle = \begin{pmatrix} 3-4i & -5i \end{pmatrix} \begin{pmatrix} 3+4i \\ 5i \end{pmatrix} = 50$$

So the normalised is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{3}{5} + \frac{4}{5}i \\ i \end{pmatrix}$$

$$\langle\psi||\phi\rangle = \begin{pmatrix} 1 & 1-2i \end{pmatrix} \begin{pmatrix} 3+4i \\ 5i \end{pmatrix} = ((3+4i) + (1-2i)(5i) = 13+9i$$

$$\langle\phi||\psi\rangle = (\langle\psi||\phi\rangle)^* = 13-9i$$

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Given that the state of the system is $|\psi\rangle$ what is the probability of measuring some attributes whose eigenvector is $|\phi\rangle$?

The probability is

$$P = \frac{\langle\phi||\psi\rangle\langle\psi||\phi\rangle}{\langle\psi||\psi\rangle\langle\phi||\phi\rangle} = \frac{13^2+9^2}{6\times 50} = \frac{5}{6}$$

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