Physics 200-04
Assignment 5

1)Old Quantum Mechanics: The Bohr Sommerfeld Quantum rules stated that if the classical orbit was closed (returned on itself) then the quantum rule was that

$$\int p\dot{q}dt = nh \tag{1}$$

where the configuration variable (eg the positions) is q, and p is the momentum. Assuming that the momentum is $p = mv = m\dot{q}$,(ie the dot denotes derivative with respect to time) and we are looking at a harmonic oscillator

$$m\ddot{q} = -kq \tag{2}$$

and the energy is

$$\frac{1}{2}\left(m\dot{q}^2 + kq^2\right) \tag{3}$$

What is the general solution to the equations of motion? Express the energy and the quantum integral in terms of the parameters of the general solution. Show that the quantum condition lead to the result

$$E_{harm\ osc} = nh\nu \tag{4}$$

where ν is the frequency of the oscillator $\frac{1}{2\pi} sqrt \frac{k}{m}$.

(The true quantum answer is $E = (n + \frac{1}{2})h\nu$)

The most general solution for the equation of motion of a harmonic oscillator is

$$q(t) = Acos(\sqrt{\frac{k}{m}}t) + Bsin(\sqrt{\frac{k}{m}}t) \tag{5}$$

The velocity then is

$$v(t) = \frac{dq(t)}{dt} = \sqrt{\frac{k}{m}}\left(-Asin(\sqrt{\frac{k}{m}}t) + Bcos(\sqrt{\frac{k}{m}}t)\right) \tag{6}$$

The energy of a Harmonic oscillator is $\frac{1}{2}(mv^2+kq^2)$. Thus, the energy is

$$E=\frac{1}{2}m\left(\sqrt{\frac{k}{m}}\left(-Asin(\sqrt{\frac{k}{m}}t)+Bcos(\sqrt{\frac{k}{m}}t)\right)\right)^2+\frac{1}{2}k\left(Acos(\sqrt{\frac{k}{m}}t)+Bsin(\sqrt{\frac{k}{m}}t)\right)^2=\frac{1}{2}k(A^2$$

The quantum integral is (where T is the period of oscillation)

$$=\int_0^T k(A^2sin(\sqrt{\frac{k}{m}}t)^2+B^2cos(\sqrt{\frac{k}{m}}t)^2+2ABsin(\sqrt{\frac{k}{m}}t)cos(\sqrt{\frac{k}{m}}t))dt=\int_0^T kA^2\left(\frac{1}{2}(1-cos(2(\sqrt{$$

since the integral of the cosine funtion and sin function over one whole period is zero.

But the period of the harmonic oscillator is independent of the energy for a Harmonic oscillator, and is just given by

$$T=2\pi\sqrt{\frac{m}{k}} \tag{12}$$

Thus we have

$$nh=ET=E(2\pi\sqrt{\frac{m}{k}} \tag{13}$$

$$E=n\hbar\sqrt{\frac{k}{m}} \tag{14}$$

The true quantum result is

$$E=(n+\frac{1}{2})\hbar\sqrt{\frac{k}{m}} \tag{15}$$

****

2a)Consider the earth in circular orbit around the sun. What would the be the difference in radius for two adjacent energy levels of the earth? (Remember that the quantization condition for the earth is that the momentum of the earth times its velocity integrated over time around one orbit is equal to an integer times h. In this case this integer is called the principle quantum number. I want the difference in radii corresponding to adjacent integers where one of the integers corresponds to the current orbit of the earth.) What is the energy difference between these two orbits as a fraction of the earth's orbital energy?

Hint: You need to use the various pieces of information about Newtonian orbits– Kepler's laws, Newton's laws for orbits, etc.

b)Bohr Correspondence: Show that the frequency of the emitted photon or graviton from the earth in circular orbit around the sun from the decay between two adjacent principle quantum numbers for the earth approximately equals the orbital frequency of the earth.

Ie, show from the first part that the energy is proportional to $\frac{K}{n^2}$ where K is a constant expressed in terms of the masses of the earth and sun, and the Newtonian Gravitational constant. Then show that the derivative of E with respect to n ($E_n - E_{n-1} = \frac{E_n - E_{n-1}}{(\Delta n=1)} \approx \frac{dE_n}{dn}$) divided by h is just the orbital frequency for the current radius of the earth's orbit.

Bohr's correspondence principle basically says that at large quantum numbers the quantum system should behave like a classical system. Since for a classical system one would expect the frequency of the emitted radiation to equal the orbital frequency, the quantum frequency of transition between adjacent levels should approximately be the classical frequency.

---

[6]– [3]on each part. [1] for numerical

This is just the same as the orbit of the electron around the nucleus, except that the constants change. We have

$$\frac{mv^2}{R} = \frac{GMm}{R^2} \tag{16}$$

as the acceleration/force balance equation. We also have the energy equation

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R} \tag{17}$$

Finally we have the quantizaton condition

$$nh = \int pvdt = \int mv^2 dt = mv^2 T \tag{18}$$

where T is the period. We also have that $\frac{v=2\pi R}{T}$. Thus

$$nh = \int m\frac{(2\pi R)^2}{T} \tag{19}$$

and from the acceleration equation

$$m(2\pi)^2\frac{R^3}{T^2} = GMm \tag{20}$$

Solving the first for T and substituting into the second, we have

$$m(1\pi)^2R^3\left(\frac{nh}{m(2\pi R)^2}\right)^2 = GMm \tag{21}$$

or

$$R = n^2\frac{h^2}{(2\pi)^2m^2GM} \tag{22}$$

or

$$\begin{aligned} n^2 &= \frac{(2\pi)^2m^2GMR}{h^2} \\ &\approx \frac{400(3\times10^{24}kg)^2 6.6\times10^-112\times10^{30}kg1.4\times10^12m}{(6.6\times10^-34)^2} \end{aligned}$$
$$= 1.5\times10^{149} \quad (23)$$

or

$$n \approx 3.8\times10^{74} \tag{24}$$

The difference in radii between n and $n-1$ is

$$R_n - R_in - 1 = \frac{h^2}{(2\pi)^2m^2GM}(n^2-(n-1)^2) = \frac{h^2}{(2\pi)^2m^2GM}(2n-1) \approx \frac{2R}{n}(25)$$

Thus

$$R_n - R_{n-1} \approx 2 \quad \frac{1.4\times10^12m}{3.8\times10^{74}} = .73\times10^{-62}m \tag{26}$$

This is a very very very very small difference.

b) From part 1, we can also solve for the energy. Following the notes for the energy of the H atom, we find that

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{GMm}{2R} \tag{27}$$

using the acceleration balance equation. But from the above we know what R is, and thus find

$$E = \frac{(2\pi)^2 G^2 M^2 m^3}{h^2 n^2} \tag{28}$$

The difference in energy between two levels is

$$\begin{aligned} E_n - E_{n-1} &= -\frac{(2\pi)^2 G^2 M^2 m^3}{h^2}\left(\frac{1}{n^2} - \frac{1}{(n-1)^2}\right) \\ &= \frac{(2\pi)^2 G^2 M^2 m^3}{h^2}\left(\frac{n^2-(n-1)^2}{n^2(n-1)^2}\right) \\ &\approx \frac{(2\pi)^2 G^2 M^2 m^3}{h^2}\frac{2}{n^3} \end{aligned} \tag{29}$$

The frequency is

$$\nu = \frac{\Delta E}{h} = \frac{(2\pi)^2 G^2 M^2 m^3}{h^3}\frac{2}{n^3} = 3 \times 10^-8/sec \tag{30}$$

or the period is one year as one would expect.

One can show this explicitly, and easily. E goes as $\frac{1}{n^2}$ while T goes as n^3 (T^2 is proportional to R^3 and since R goes as n^2, T goes as n^3.

Now,

$$nh = \int mv^2 dt = \int 2KEdt \tag{31}$$

But, by the virial theory for the inverse r power law potential, or from the acceleration balance equation, the $PE = -2KE$ so, that $KE = -E$. This total energy is $E = KE + PE = -KE$. This gives the

$$nh = -\int 2Edt = 2ET \tag{32}$$

Differentiating both sides by n and recalling the power law behaviour of E and T we have

$$h = -2\frac{dE}{dn}T - 2E\frac{dT}{dn} = +2\frac{2E}{n}T - 2E\frac{3T}{n} = \frac{-2E}{n}T = \frac{dE}{dn}T \tag{33}$$

Thus we have

$$\frac{dE}{dn} = \frac{h}{T} \tag{34}$$

But $\frac{dE}{dn}$ is approximately $E_n - E_{n-1}$ for large n. Thus we immediately get that

$$\Delta E \approx \frac{h}{T} \tag{35}$$

The radiation emitted in making a transition from the n^{th} orbit to the $(n-1)^th$ orbit has just the frequency of the orbit, as one might expect.

****

3. Calculate the following complex operations: i) $(3-4i)((1-5i)-(2+i))$

$$(3-4i)(-1-6i) = (-3-24-18i+4i) = 27-14i$$

****

ii) $(10-5i)(2+2i)/(3+4i)$

$$\frac{(10-5i)(2+2i)(3-4i)}{(3+4i)(3-4i)} = \frac{10}{25}(2-i)(1+i)(3-4i) = \frac{2}{5}(3+i)(3-4i) = \frac{2}{5}(13-9i) = \frac{26}{5} - \frac{19}{5}i$$

****

iii)Find the roots of the following equation, using complex numbers if necessary

$$x^2 + 3x + 5 = 0 \tag{36}$$

$$x = \frac{-3 \pm \sqrt{9 - 4 \cdot 1 \cdot 5}}{2} = -\frac{3}{2} \pm \frac{\sqrt{11}}{2}i$$

****

iv) Expand the expression $(x+2i)(x+1+i)/(x+i)$ into a single expression $a+bi$ where both a and b are real numbers. Assume x is a real number.

$$(x+2i)(x+1+i)/(x+i) = \frac{(x+2i)(x+1+i)(x-i)}{(x+i)(x-i)} = \frac{x^3+(1+2i)x^2+(1+i)x+(2+2i)}{x^2+1}$$

****

4. Multiply the matrices

i)

$$\begin{pmatrix} 1+i & 1+i \\ 1-i & 1-i \end{pmatrix} \begin{pmatrix} -1 & i \\ -i & +1 \end{pmatrix} \tag{37}$$

$$\begin{pmatrix} 1+i & 1+i \\ 1-i & 1-i \end{pmatrix} \begin{pmatrix} -1 & i \\ -i & +1 \end{pmatrix} = \begin{pmatrix} -1-i & 1+i \\ 1+i & 3-i \end{pmatrix}$$

****

ii) Find the transpose, the Hermetian transpose, and the inverse of the matrix

$$\begin{pmatrix} 1+i & 1+i \\ 1-i & 1+i \end{pmatrix} \tag{38}$$

$$\begin{pmatrix} 1+i & 1+i \\ 1-i & 1+i \end{pmatrix}^T = \begin{pmatrix} 1+i & 1-i \\ 1+i & 1+i \end{pmatrix} \tag{39}$$

$$\begin{pmatrix} 1+i & 1+i \\ 1-i & 1+i \end{pmatrix}^\dagger = \begin{pmatrix} 1-i & 1+i \\ 1-i & 1-i \end{pmatrix} \tag{40}$$

$$\tag{41}$$

The inverse is the matrix which, when multipied by the above matrix gives teh identity matrix.

$$\begin{pmatrix} 1+i & 1+i \\ 1-i & 1+i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{42}$$

or

$$(1+i)a+(1+i)c = 1 \tag{43}$$

$$(1+i)b+(1+i)d = 0 \tag{44}$$

$$(1-i)a+(1+i)c = 0 \tag{45}$$

$$(1-i)b+(1+i)d = 1 \tag{46}$$

Subtracting the third from the first, $2ia = 1$ or $a = \frac{1}{2i} = -\frac{1}{2}i$ and $c = \frac{(1-i)i}{2(1+i)} = \frac{1}{2}$.

Subtracting the fourth from the second, $b = \frac{1}{2}i$ and $d = -\frac{1}{2}i$

Thus teh inverse is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} i & i \\ 1 & -i \end{pmatrix}$$